Lecture XLI : §12.3 (cont.) Other indeterminate forms
§12.4 : Improper integrals

Recall: If f, g cont & diff'ble near $x=a$ & $g'(x)\neq 0$ for $x\neq a$ near a.

L'Hop Rule

- If $\lim\limits_{x\to a} \frac{f(x)}{g(x)} \sim \frac{0}{0}$, then $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$ provided (RHL) limit exists
- If ______ $\sim \frac{\infty}{\infty}$ ______

Same applies for a in $\mathbb{R}$ or $a=\pm\infty$.

TODAY: Other indeterminacies!

§1. $0.\infty$

$0.\infty$ (f g) become $\to \frac{\infty}{\infty}$ as $g/(1/f)$, $\to \frac{0}{0}$ as $f/(1/g)$

Algebra: Divide by reciprocal of one factor

Ex 1: $\lim\limits_{x\to 0^+} \underset{\substack{\downarrow\\ 0}}{x}\, \underset{\substack{\downarrow\\ -\infty}}{\ln x} = \lim\limits_{x\to 0^+} \underbrace{\frac{\ln x}{1/x}}_{\infty/\infty} \underset{\text{L'Hop}}{=} \lim\limits_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim\limits_{x\to 0^+} -x = \boxed{0}$

Other option: $\lim\limits_{x\to 0^+} \underbrace{\frac{x}{1/\ln x}}_{0/0} = \lim\limits_{x\to 0^+} \frac{1}{\frac{-1}{x\ln^2 x}} = \lim\limits_{x\to 0^+} -x\ln^2 x$ is worse!

Upshot: Reciprocal chosen should have ~~easy~~ simpler derivative.

§2. 3 cases: 0^0, ∞^0, 1^∞

Algebra: Take $\ln$ & ~~then~~ exp. (0^0, ∞^0, 1^∞)

$\ln\left(\lim\limits_{x\to a} F(x)\right) = \lim\limits_{x\to a} \ln F(x)$ provided $F(x)>0$ near $x=a$.

$\exp\left(\text{———}\right) = \lim\limits_{x\to a} e^{F(x)}$

Ex 2: $\lim\limits_{x\to 0^+} x^x = ?$ [0^0 type] Take $\ln$!

$\lim\limits_{x\to 0^+} \ln(x^x) = \lim\limits_{x\to 0^+} \underset{0\ \infty}{x\ln x} \underset{\text{Ex 1}}{=} 0$ so $\lim\limits_{x\to 0^+} x^x = e^0 = \boxed{1}$

Ex 3: $\lim_{x\to\infty} x^{1/x} = ?$ $[\infty^0 \text{ type}]$ Take ln!

$$\lim_{x\to\infty} \ln(\underbrace{x^{1/x}}_{>0}) = \lim_{x\to\infty} \frac{1}{x}\ln x = \lim_{x\to\infty} \underbrace{\frac{\ln x}{x}}_{\frac{\infty}{\infty}} \underset{\text{L'Hop}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \boxed{0}$$

So $\lim_{x\to\infty} x^{1/x} = e^0 = \boxed{1}$

Ex 5 $\lim_{x\to 0} (1+ax)^{1/x} = ?$ for all $a \neq 0$ $[1^\infty \text{ Type}]$ Take ln

$$\lim_{x\to 0} \ln\big(\underbrace{(1+ax)^{1/x}}_{>0}\big) = \lim_{x\to 0} \underset{\downarrow \pm\infty}{\frac{1}{x}} \underset{\downarrow 0}{\ln(1+ax)} = \lim_{x\to 0} \underbrace{\frac{\ln(1+ax)}{x}}_{0/0} \underset{\text{L'Hop}}{=} \lim_{x\to 0} \frac{\frac{a}{1+ax}}{1} = a$$

So $\lim_{x\to 0} (1+ax)^{1/x} = \boxed{e^a}$

§3 $\infty - \infty$:

Algebra: Take common denominator

Ex 6: $\lim_{x\to\frac{\pi}{2}} \underset{\downarrow \pm\infty}{\sec c} - \underset{\downarrow \pm\infty}{\tan x} = \lim_{x\to\pi/2} \frac{1}{\cos x} - \frac{\sin x}{\cos x} = \lim_{x\to\pi/2} \underbrace{\frac{1-\sin x}{\cos x}}_{0/0} \underset{\text{L'Hop}}{=} \lim_{x\to\frac{\pi}{2}} \frac{-\cos x}{-\sin x}$

$= \frac{0}{1} = \boxed{0}$

⚠ Sometimes direct algebraic manipulations are faster than L'Hopital!

Ex 1: $\lim_{x\to 0} \underbrace{\frac{\sin^4 x}{x^4}}_{0/0} \underset{y^4 \text{ cont}}{=} \left(\lim_{x\to 0} \frac{\sin x}{x}\right)^4 = 1^4 = \boxed{1}$ (It would have required 4 L'Hop iterations)

Ex 2: $\lim_{x\to\infty} \underbrace{\frac{\sqrt{x^2+1}}{x}}_{\infty/\infty} = \lim_{x\to\infty} \sqrt{\frac{x^2+1}{x^2}} = \lim_{x\to\infty} \sqrt{1+\underset{\to 0}{\frac{1}{x^2}}} = \sqrt{1+0} = \boxed{1}.$

L'Hop? $L = \lim_{x\to\infty} \frac{\frac{x}{\sqrt{x^2+1}}}{1} = \lim_{x\to\infty} \frac{1}{\frac{\sqrt{x^2+1}}{x}} = \frac{1}{L}$ so we haven't improved much ($L^2=1$ so $L=\pm 1$ if it exists!)

Ex 3: Rat'l Functions : easier to factor & divide by large power of x:

$\frac{\infty}{\infty} \sim \lim_{x\to\infty} \frac{3x^4+x}{x^4+2x^2+1} = \lim_{x\to\infty} \frac{3+1/x^3}{1+2/x^2+1/x^4} = 3$ vs 4 iterations of L'Hop.

§4 Improper integrals

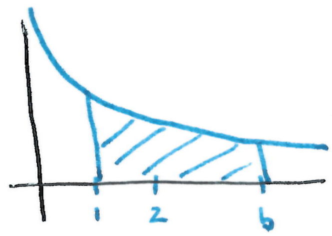
GOAL: Compute $\int_a^b f(t)\,dt$ when (A) $a = -\infty$ and/or $b = +\infty$

(B) $\lim_{t \to a^+} f(t) = -\infty$ and/or $\lim_{t \to b^-} f(t) = +\infty$

(asymptotes)

Examples for (A)

① Calculate the area under the curve $y = \frac{1}{x^2}$ with $x \geq 1$

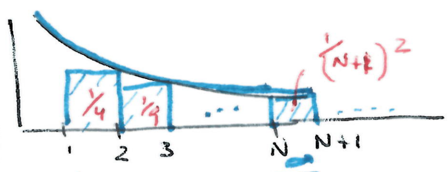


$$\int_1^b \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_1^b = 1 - \frac{1}{b} \xrightarrow[b \to \infty]{} 1 - 0 = \boxed{1}$$

$$\text{Area} = \lim_{b \to \infty} \int_1^b \frac{1}{x^2}\,dx = 1 \underset{\text{define}}{=:} \int_1^\infty \frac{1}{x^2}\,dx$$

We say the improper integral converges

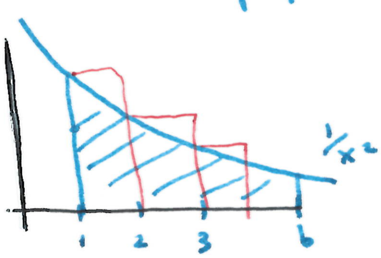
Note Using Riemann Sums $x_n = \frac{1}{n+1}$



so $1 = \text{Area} \geq \frac{1}{4} + \frac{1}{9} + \cdots = \lim_{N \to \infty} \sum_{n=2}^{N} \frac{1}{n^2} \underset{\text{def}}{=} \sum_{n=2}^{\infty} \frac{1}{n^2}$ series

We'll use improper integrals to study convergence of series. (Ch 13)

②



Area under $y = \frac{1}{x}$ with $x \geq 1$?

$$\int_1^b \frac{1}{x}\,dx = \ln x \Big|_1^b = \ln b - \ln 1 = \ln b \xrightarrow[b \to \infty]{} \infty$$

The integral diverges!

Note $\text{Area} \leq 1 + \frac{1}{2} + \frac{1}{3} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n}$ so series will diverge

③ In general: $\int_1^b \frac{dx}{x^p} = \frac{x^{1-p}}{1-p}\Big|_1^b = \frac{1}{1-p}\left(b^{1-p} - 1\right)$

$p \geq 0 \quad p \neq 1$

$$\int_1^\infty \frac{dx}{x^p} = \lim_{b \to \infty} \int_1^b \frac{dx}{x^p} = \begin{cases} \infty & \text{if } p < 1 \quad (b^{1-p} \to \infty) \\ \frac{1}{p-1} & \text{if } p > 1 \quad \left(b^{1-p} = \frac{1}{b^{p-1}} \to \frac{1}{\infty} = 0\right) \end{cases}$$

Definition (A) $\int_a^\infty f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx$

Applications: (1) Convergence / divergence of power series (Ch 13)

(2) Laplace Transform: $L_f(p) = \int_0^\infty e^{-px} f(x)\,dx$ for a function f

(3) GAMMA function: $\Gamma(p) = \int_0^\infty e^{-x} \underbrace{x^{p-1}\,dx}_{dv} \underset{\text{parts}}{=} \lim_{b\to\infty} \frac{e^{-x}x^p}{p}\Big|_0^b + \int_0^\infty \frac{x^p e^{-x}}{p}\,dx$

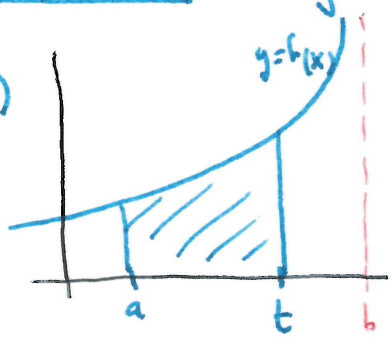
$v = \frac{x^p}{p}$

$$= \frac{1}{p}\lim_{b\to\infty} \frac{x^p}{e^x} + \frac{1}{p}\Gamma(p+1) = \frac{\Gamma(p+1)}{p}$$

So $\boxed{\Gamma(p+1) = p\,\Gamma(p) \text{ for any } p \neq 0}$ ($\Gamma(p)$ interpolates $(p-1)!$ for $p \geq 1$)

Examples for (B)? Say $f(x)$ has a vertical asymptote at $x=b$.

Def (B)



Area under the curve $= \int_a^b f(x)\,dx$

$$= \lim_{t\to b^-} \int_a^t f(x)\,dx$$

Note: This limit may or may not exist!

If it exist, we say the integral converges

If not or $=\infty$, ________ diverges

Ex 1: $\int_0^1 \frac{dx}{\sqrt{1-x}} = \lim_{t\to 1^-} \int_0^t \frac{dx}{\sqrt{1-x}} = \lim_{t\to 1^-} -2\sqrt{1-x}\,\Big|_0^t = \lim_{t\to 1^-} 2-2\sqrt{1-t} = \boxed{2}$ converges

Ex 2 $\int_0^1 \frac{dx}{1-x} = \lim_{t\to 1^-} \int_0^t \frac{dx}{1-x} = \lim_{t\to 1^-} -\ln(1-x)\,\Big|_0^t = \lim_{t\to 1^-} -\ln(1-t) = \boxed{\infty}$ diverges

- Same idea is used to define $\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$ if $x=a$ is vert asympt!

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to -\infty} \int_t^b f(x)\,dx$$

Q: What if we have issues at both a & b?

A: Use $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ for ANY $a<c<b$

The result should be independent of our choice of c (1st 2 integrals on RHS) will be improper).

Typical choice: $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$

Ex $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$

$$= \lim_{t\to-\infty} \int_{t}^{0} \frac{dx}{1+x^2} + \lim_{t\to+\infty} \int_{0}^{t} \frac{dx}{1+x^2}$$

$$= \lim_{t\to-\infty} \arctan(x)\Big|_{t}^{0} + \lim_{t\to+\infty} \arctan(x)\Big|_{0}^{t}$$

$$= 0 - \left(-\frac{\pi}{2}\right) + \left(\frac{\pi}{2} - 0\right) = \boxed{\pi}$$