Lecture XLII: § 13.1 What is an infinite series?

Def An infinite series (of constants), or series is an expression of the form

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n \qquad (*)$$

- a_n is called the n^{th} term and it's usually given by a simple formula

Eg: $a_n = \frac{1}{2^n}$, so $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n}$.

GOAL: Make sense of $(*)$ both formally & exactly (ie can we compute its value or determine the value is ∞ or does not exist).

Example 1 (Infinite decimals):

$$0.a_1a_2a_3a_4\cdots = \frac{a_1}{10} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \cdots + \frac{a_n}{10^n} \qquad a_i \in \{0,1,\ldots,9\} \text{ for all } i$$

Eg (1) $\frac{1}{3} = 0.333\cdots$ means $\frac{1}{3} = \frac{3}{10} + \frac{3}{10^2} + \cdots = \sum_{n=1}^{\infty} \frac{3}{10^n} = 3\sum_{n=1}^{\infty} \frac{1}{10^n}$

so $\frac{1}{9} = \sum_{n=1}^{\infty} \frac{1}{10^n} = \sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n$

(2) $\pi = 3.1415\cdots \rightsquigarrow \pi = 3 + \sum \frac{a_n}{10^n}$ $\quad a_1 = 1,\ a_2 = 4,\ a_3 = 1, \ldots$

Example 2 (Geometric series)

Q: Can we write $\frac{1}{1-x}$ as a series in x?

A: "YES" via long division, where we take x, x^n as the leading terms of $1-x$ & $1-x^n$.

$$\begin{array}{r|l} & x^{n-1} + x^{n-2} + \cdots + x + 1 \\ \hline 1-x & 1 - x^n \\ & x^{n-1} - x^n \\ \hline & 1 - x^{n-1} \\ & \vdots \end{array}$$

so $1 - x^n = (1-x)(1 + \cdots + x^{n-1})$

We get $\frac{1}{1-x} - \frac{x^n}{1-x} = 1 + \cdots + x^{n-1}$

$$\boxed{\frac{1}{1-x} = 1 + \cdots + x^{n-1} + \frac{x^n}{1-x}}$$

We can continue this process as $n \to \infty$. If $x^n \xrightarrow[n\to\infty]{} 0$ (which happens if $|x| < 1$)

we get $\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$. "Power series expansion of $\frac{1}{1-x}$"

Check: $x = \frac{1}{10}$ $\quad \frac{1}{1 - 1/10} = \frac{10}{9} \stackrel{?}{=} 1 + \sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n$ agrees with $\frac{1}{9} = \frac{10}{9} - 1 = \sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n$.

Variants: replace x by $(-x)$, or (x^2).

- $\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 + (-x) + (-x)^2 + \cdots = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$

  $1 - x + x^2 - x^3 + \cdots$

- $\frac{1}{1-x^2} = 1 + (x^2) + (x^2)^2 + \cdots = \sum_{n=0}^{\infty} x^{2n}$; • $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$.

Q: Can we manipulate these power series as if they were infinite polynomials?
eg: Term-by-term integration.

① $\int \frac{dx}{1+x} = \ln(1+x) = \int \sum_{n=0}^{\infty} (-1)^n x^n \, dx \overset{?}{\underset{\text{SWAP}}{=}} \sum_{n=0}^{\infty} \underbrace{\int (-1)^n x^n \, dx}_{= (-1)^n \frac{x^{n+1}}{n+1}} =$

$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \underset{\substack{m = n+1 \geq 1 \\ n = m-1}}{=} \sum_{m=1}^{\infty} (-1)^{m-1} \frac{x^m}{m} = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots$

We get an expression for $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots$ (provided we can swap $\int$ & $\sum_{n=0}^{\infty}$.)

② $\tan^{-1}(x) = \int \frac{1}{1+x^2} dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} dx \overset{?}{\underset{\text{SWAP}}{=}} \sum_{n=0}^{\infty} \int (-1)^n x^{2n} dx$

$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \underset{m=2n+1}{=} x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

So we get an expression for $\tan^{-1}(x)$ as a power series again provided we can swap $\int$ & $\sum_{n=0}^{\infty}$.

Q: Once we've established these identities, can we evaluate at some x?

- Set $x=1$ in ① (which may not work since $\frac{1}{1+x} = \sum (-x)^n$ only holds if $|x|<1$)
  to guess $\ln 2 = \ln(1+1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$
- Set $x=1$ in ② (again, this is not nec. an allowed evaluation) to
  guess $\frac{\pi}{4} = \tan^{-1}(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

Our guesses, in particular, will say that the power series on the right, if they converge, should have the above values. But the guesses may fail to be true.

Example 3 (Differential Equations)

IDEA: Given a differential equation: ($y' = y$, $y'' + a^2 y = 0$, etc.) can we guess a solution by writing it as $y(x) = \sum_{n=0}^{\infty} a_n x^n$, then differentiating term-by-term & writing equalities between coefficients?

Ex 1: $y' = y$ $\rightsquigarrow$

$$y = \boxed{a_0} + a_1 x + \cdots + a_n x^n + \cdots$$

$$y' = a_1 + a_2 2x + \cdots + a_n n x^{n-1} + \cdots$$

$$\boxed{a_1} + 2a_2 x + \cdots + a_{n+1}(n+1) x^n + \cdots$$

$\rightsquigarrow$ equating both series means term-by-term equalities.

$$a_0 = a_1, \quad a_1 = 2a_2, \quad \ldots, \quad a_n = (n+1)a_{n+1} \text{ for all } n \geq 0$$

We get a recursion. $a_{n+1} = \dfrac{a_n}{n+1}$, can pick any a_0 to begin.

$$a_{n+1} = \frac{a_n}{n+1} = \frac{1}{n+1}\frac{a_{n-1}}{n} = \frac{1}{n+1}\frac{1}{n}\frac{a_{n-2}}{n-1} = \cdots = \frac{1}{(n+1)!}a_0$$

$$y = a_0\left(1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) \qquad y(0) = a_0$$

But we know that $y = \lambda e^x$ is the general form of the solutions! $\lambda = y(0)$

We get $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \dfrac{x^n}{n!}$ ($0! = 1$ by convention).

Ex 2: $y'' + y = 0$ Has general soln $= c_1 \sin x + c_2 \cos x$
$= A \sin(x + b)$.

Use power series expansion: $y'' = 2a_2 + 3\cdot 2 a_3 x + \cdots + a_{n+1}(n+1)n\, x^{n-1} + \cdots$

To get:

$a_0 + 2a_2 = 0$

$a_1 + 3\cdot 2 a_3 = 0$ $\rightsquigarrow$ $a_{n+1}(n+1)n + a_{n-1} = 0$ for all $n \geq 1$

$a_2 + 4\cdot 3\, a_4 = 0$ Equivalently: $\boxed{a_{n+2}(n+2)(n+1) = -a_n \text{ for all } n}$

$\vdots$

Break the sum into even and odd powers (can we rearrange a series freely??)

to get $y = \sum_{n=0}^{\infty} a_n x^n \overset{?}{=} \sum_{n \text{ odd}} a_n x^n + \sum_{n \text{ even}} a_n x^n$

We get $a_n = -\dfrac{a_{n-2}}{n(n-1)} = \dfrac{-1}{n(n-1)}\dfrac{(-a_{n-4})}{(n-2)(n-3)} = \dfrac{a_{n-4}}{n(n-1)(n-2)(n-3)}$ $\rightsquigarrow$ $\begin{cases} a_{2k} = \dfrac{(-1)^k}{(2k)!} a_0 \\ a_{2k+1} = \dfrac{(-1)^k}{(2k+1)!} a_1 \end{cases}$

$$y = a_0 \underbrace{\boxed{\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}}}_{= \sin(x)} + a_1 \underbrace{\boxed{\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}}}_{= \cos(x)} = a_0 \sin x + a_1 \cos x$$