Lecture XLIII : §13.2 Convergent sequences

Write $\sum_{n=0}^{\infty} a_n$ as being approximated by partial sums $S_1 = a_1$
$S_2 = a_1 + a_2$
$\vdots$
$S_n = a_1 + \cdots + a_n$

So $\sum_{n=0}^{\infty} a_n = \lim_{n\to\infty} \boxed{S_n}$ → sequence.

Example (geometric series) $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 2 = \lim \underbrace{1 + \frac{1}{2} + \cdots + \frac{1}{2^n}}_{S_n}$

2 is the limit of the sequence of partial sums.

Def: A sequence is an infinite list of numbers, indexed by the natural numbers, Write $\{x_n\}_{n\in\mathbb{N}} = x_1, x_2, x_3, \ldots, x_n, x_{n+1}, \ldots\ldots$

• x_n is called the n^{th} term of the sequence.

Examples:

		BOUNDED?	INC/DEC?
(a) $x_n = 1$ for all n gives	$1, 1, \ldots\ldots$ = constant sequence	$A = B = 1$	const, so incr & decr.
(b) $x_n = \frac{1-(-1)^n}{2}$ "	$1, 0, 1, \ldots$	$A=0,\ B=1$	none
(c) $x_n = \frac{n-1}{n}$ "	$0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$	$A=0,\ B=1$	str. incr.
(d) $x_n = \frac{(-1)^{n+1}}{n}$ "	$1, \frac{-1}{2}, \frac{1}{3}, \frac{-1}{4} \ldots$	$A=-1,\ B=1$	none
(e) $x_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n}$	$= \frac{-\left(\frac{1}{2}\right)^{n+1} + 1}{1 - \frac{1}{2}} = 2\left(1 - \left(\frac{1}{2}\right)^{n+1}\right)$ ↑ Last time $\left[\frac{1}{1-x} = 1 + \cdots + x^n + \frac{x^{n+1}}{1-x}\right]$	$A=0,\ B=2$	str. incr.
(f) $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$		$A=0,\ B=??$ none	incr.
(g) $x_n = \left(1 + \frac{1}{n}\right)^n$		$A=0,\ B=??$	??
(h) $x_n = n^{th}$ digit in the decimal exp. of π		$A=0,\ B=9$	??

Def: • A sequence $\{x_n\}$ is bounded if we can find constants A, B which satisfy $A \le x_n \le B$ for all n in $\mathbb{N}$.

• A sequence $\{x_n\}$ is increasing if $x_n \le x_{n+1}$ for all $n (\ge n_0)$

——————— str. ——— if $x_n < x_{n+1}$ for all $n\ (\ge n_0)$

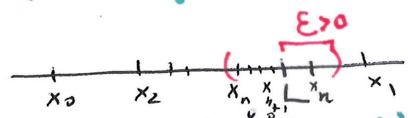
• ——————— decreasing if $x_n \ge x_{n+1}$ for all $n\ (\ge n_0)$

——————— str. ——— if $x_n > x_{n+1}$ for all $n\ (\ge n_0)$

[We can find n_0, st. the sequence $\{x_n\}_{n\ge n_0}$ (disregarding first few terms) is incr, str incr, decr. or str. incr.]

GOAL: Understand "long term behavior" of sequences (ie let $n \to \infty$)

Heuristic: $\lim_{n\to\infty} x_n = L$ means as n gets large, the value of x_n gets close to L, meaning $|x_n - L|$ gets close to 0.



Def: $\lim_{n\to\infty} x_n = L$ if for every $\varepsilon > 0$ there exists N_0 in $\mathbb{N}$ so that if $n \geq N_0$, then $|x_n - L| < \varepsilon$.

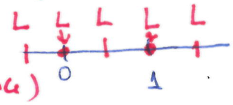
Def: A sequence $\{x_n\}$ is <u>convergent</u> if it has a limit L in $\mathbb{R}$, otherwise, we say it's <u>divergent</u>.

• <u>Techniques</u>: ① manipulate sequences algebraically (just as we did with functions)

<u>Back to examples</u>

(a) $x_n = 1$ so limit $= 1$. $(N_0 = 0)$ $|x_n - 1| = |0| < \varepsilon$ for any n.

(b) $\{x_n\} = 1, 0, 1$ limit does not exist! The sequence jumps between 2 values (oscillating sequence)



(c) $x_n = \frac{n-1}{n} = 1 - \frac{1}{n} \xrightarrow[n\to\infty]{} 1 - 0 = 1$ (given $\varepsilon > 0$ pick n_0 with $\frac{1}{n_0} < \varepsilon$ so $\frac{1}{\varepsilon} < n_0$; $1 + [\frac{1}{\varepsilon}] = n_0$ will do)

(d) $x_n = \frac{(-1)^{n+1}}{n}$ $|x_n| = \frac{1}{n} \xrightarrow[n\to\infty]{} 0$ so $x_n \xrightarrow[n\to\infty]{} 0$. same $n_0 = [\frac{1}{\varepsilon}] + 1$

(e) $x_n = 1 + \frac{1}{2} + \dots + \frac{1}{2^n} = 2\left(1 - \frac{1}{2^{n+1}}\right) \xrightarrow[n\to\infty]{} 2(1-0) = 2$
$= 2 - \frac{1}{2^n}$

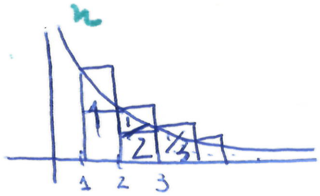
given $\varepsilon > 0$, want to find n_0 so that $|x_n - 2| = \frac{1}{2^n} < \varepsilon$ for $n \geq n_0$

$n_0 = \left[\frac{1}{\ln 2} \ln \frac{1}{\varepsilon}\right] + 1$ will do

(given $\varepsilon > 0$
$|x_n - 2| = |\frac{1}{2^n}| < \varepsilon$
if $\frac{1}{\varepsilon} < 2^n$? Take ln
$\ln \frac{1}{\varepsilon} < n \ln 2$
$\frac{1}{\ln 2} \ln \frac{1}{\varepsilon} < n$)

(f) $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$

Claim: limit $= \infty$



$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \geq \int_1^\infty \frac{1}{x}\,dx = \infty$

(g) $x_n = \left(1 + \frac{1}{n}\right)^n \xrightarrow[n\to\infty]{} e$ (future lecture) [Knows $(1+x)^{1/x} \xrightarrow[x\to 0^+]{} e$ Take $x = \frac{1}{n}$.]

(h) $x_n = n^{\text{th}}$ digit of π is divergent (later)