Lecture XLIV: §13.2 (cont.) Convergent series

Recall: $\lim\limits_{n\to\infty} x_n = L$ if for every $\varepsilon > 0$, we can find N_0 in $\mathbb{N}$ so that if $n \geq N_0$ we have $|x_n - L| < \varepsilon$.

Last time: many examples, but what are the main techniques?

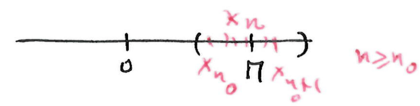
§1. Limit laws

Prop If $\lim\limits_{n\to\infty} x_n = L$ & $\lim\limits_{n\to\infty} y_n = M$, then the sequences $\{x_n \pm y_n\}_{n\in\mathbb{N}}$ & $\{x_n \cdot y_n\}_{n\in\mathbb{N}}$ are convergent & $\lim\limits_{n\to\infty}(x_n \pm y_n) = L \pm M$

$\lim\limits_{n\to\infty}(x_n \cdot y_n) = LM$

Furthermore, if $M \neq 0$, the sequences $\left\{\frac{x_n}{y_n}\right\}_{n\geq n_0}$ converges & $\lim\limits_{n\to\infty} \frac{x_n}{y_n} = \frac{L}{M}$.

Proof: Variant of ε/δ games for Limit laws of functions. Note: first few terms of $\{y_n\}$ can be zero, but if $M \neq 0$, we know eventually they are all $y_n \neq 0$ $\forall n \geq n_0$. (after some n_0)



Example: • $z_n = \dfrac{n^2+4}{5n^2+6n+7} \underset{\substack{\text{divide} \\ \text{by } n^2}}{=} \dfrac{\overbrace{1 + 4/n^2}^{= x_n}}{\underbrace{5 + 6 + 7/n^2}_{= y_n}} \xrightarrow[n\to\infty]{} \boxed{\frac{1}{5}}$.

• $z_n = \sqrt{n+1} - \sqrt{n} = \left(\sqrt{n+1} - \sqrt{n}\right)\dfrac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \dfrac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \dfrac{1}{\sqrt{n+1}+\sqrt{n}} \xrightarrow[n\to\infty]{} \boxed{0}$

§2. Squeeze-Thm:

Thm: Suppose 3 sequences $\{a_n\}_n$, $\{x_n\}_n$, $\{b_n\}_n$ satisfy

(1) $a_n \leq x_n \leq b_n$ for all $n \geq n_0$ ("n large enough")

(2) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = L$

Then $\{x_n\}_n$ is convergent & $\lim\limits_{n\to\infty} x_n = L$

Example (1) $x_n = \dfrac{1}{n!}$ $\quad 0 \leq x_n \leq \dfrac{1}{n}$, with $0 \xrightarrow[n\to\infty]{} 0$ and $\dfrac{1}{n} \xrightarrow[n\to\infty]{} 0$, so $x_n \xrightarrow[n\to\infty]{} 0$

(2) $x_n = \frac{a^n}{n!}$ for $a>0$ fixed. Claim: $x_n \xrightarrow[n\to\infty]{} 0$

- $0 \leq x_n$ for all n, so take $a_n = 0$ constant sequence.
- Need to find $\{b_n\}_n \xrightarrow[n\to\infty]{} 0$ with $x_n \leq b_n$ for all n large enough.

IDEA: $x_n = \frac{a}{n} \underbrace{\frac{a}{n-1} \cdots \frac{a}{2}\,\frac{a}{1}}_{=x_{n-1}}$

Pick $n_0>0$ with $\frac{a}{n_0} < \frac{1}{2}$ & write any $n \geq n_0$ as $n = n_0 + k$ for $k \geq 0$.

So $x_n = \frac{a^n}{n} = \frac{a^{n_0+k}}{(n_0+k)!} = \frac{a^{n_0}}{n_0!}\,\frac{a}{n_0+1}\,\frac{a}{n_0+2}\cdots\frac{a}{n_0+k} < \frac{a^{n_0}}{n_0!}\left(\frac{a}{n_0}\right)^k < \frac{a^{n_0}}{n_0!\,2^k}$

$<\frac{a}{n_0}$ .... $<\frac{a}{n_0}$ $<\frac{1}{2}$

Set $b_n = \begin{cases} \frac{a^{n_0}}{n_0!\,2^{n-n_0}} & \text{for } n \geq n_0 \\ 0 & \text{otherwise} \end{cases}$ $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \boxed{\frac{a^{n_0}}{n_0!}}\,\frac{1}{2^{n-n_0}} = 0$

FIXED $\quad 2^{n-n_0} \xrightarrow[n\to\infty]{} \infty$

Then $x_n \leq b_n$ for $n \geq n_0$. & by Squeeze Thm $\lim_{n\to\infty} x_n = 0$.

Note: y_n grows for a while if a is large, but then it starts decreasing, right after $\frac{a}{n} < 1$.

§3 Convergence Criteria

THM 1: Assume $\{x_n\}$ is increasing ($x_n \leq x_{n+1}$ for all n large enough). Then $\{x_n\}$ is convergent if and only if ($\Longleftrightarrow$) $\{x_n\}$ is bounded (from above)

THM 2: Assume $\{y_n\}$ is decreasing ($y_n \geq y_n$ for all n large enough). Then $\{y_n\}$ is convergent if and only if $\{y_n\}$ is bounded (from below)

- Observation: Need only show THM 1. To show THM 2, given y_n, we set $x_n = -y_n$. Then x_n will be increasing & x_n bounded $\Longleftrightarrow$ y_n bounded

$\Updownarrow$ THM 1.

x_n convergent $\Longleftrightarrow$ y_n convergent

- To prove THM 1, we need to check both implication. The direction ($\Longrightarrow$) holds in general, so we write it as a separate lemma.

Lemma: If $\{x_n\}_n$ is convergent, then $\{x_n\}_n$ is bounded.

Proof: Write $\lim_{n\to\infty} x_n = L$. For $\varepsilon = 1$, we can find n_0 such that $|x_n - L| < 1 = \varepsilon$ for all $n \geq n_0$, meaning $L-1 < x_n < L+1$ for $n \geq n_0$.

For $n < n_0$, we need a different bound:

Pick $\left.\begin{array}{l} M = \max\{x_1, x_2, \dots, x_{n_0-1}\} \\ N = \min\{\text{\_\_\_\_\_\_\_\_\_\_}\} \end{array}\right\} \rightsquigarrow M \leq x_n \leq N$ for $n = 1, \dots, n_0 - 1$

Take $A = \min\{N, L-1\}$, $B = \max\{M, L+1\}$. Then $A \leq x_n \leq B$ for all n, so $\{x_n\}$ is bounded. $\square$

Proof of Thm 1: By double implication.

($\Rightarrow$) Is the statement of the Lemma.

($\Leftarrow$) Assume $\{x_n\}$ is bounded, we want to find the limit.

- Pick B ~~with~~ ~~[illegible]~~ $B \geq x_n$ for all n
- The smaller the B, the better the bound

We can find a least upper bound (L.U.B.) on $\mathbb{R}$ by the way $\mathbb{R}$ is constructed:

- **Least Upper Bound Axiom for $\mathbb{R}$**: every non-empty set S in $\mathbb{R}$ that has an upper bound also has a least upper bound $(= \inf\{B \text{ in } \mathbb{R} : x \leq B \text{ for all } x \text{ in } S\})$

  ↳ tightest upper bound

  - $\mathbb{Q}$ does not have this property: $S = \{x \text{ in } \mathbb{Q} : x < \sqrt{2}\}$ has upper bounds ($M = 4$, for example), but LUB $= \sqrt{2}$ not in $\mathbb{Q}$.

Set L = least upper bound for $\{x_n\} = \{x_1, x_2, \dots\}$

Claim: $L = \lim_{n\to\infty} x_n$

Def of L.U.B. says that for any $\varepsilon > 0$, $L - \varepsilon$ is not an upper bound, so we can find n_0 with $L - \varepsilon < x_{n_0} \leq L$

But x_n is increasing, so $L - \varepsilon < x_{n_0} \leq x_n \leq L$ for all $n \geq n_0$.

Then $|x_n - L| < \varepsilon$ for all $n \geq n_0$, as we wanted.

Examples ① $x_n = \frac{\text{sen}(n!)}{3^n} \xrightarrow[n\to\infty]{} 0$ by Squeeze Thm:

$$\frac{-1}{3^n} \leq \frac{\text{sen}\, n!}{3^n} \leq \frac{1}{3^n} \qquad (-1 \leq \text{sen}\, x \leq 1)$$

$\frac{-1}{3^n} \xrightarrow[n\to\infty]{} 0$, $\frac{1}{3^n} \xrightarrow[n\to\infty]{} 0$

② $x_n = 1 - \frac{1}{3^n} \xrightarrow[n\to\infty]{} 0$ We can see this in 2 ways.

(1) Limit Laws:

$$\left.\begin{array}{l} 1 \xrightarrow[n\to\infty]{} 1 \\ \frac{1}{3^n} \xrightarrow[n\to\infty]{} 0 \end{array}\right\} \quad x_n = 1 - \frac{1}{3^n} \longrightarrow 1 - 0 = \boxed{1}$$

(2) • $x_n \leq x_{n+1}$ for all n because $\frac{1}{3^{n+1}} \leq \frac{1}{3^n}$ so $1 - \frac{1}{3^{n+1}} \geq 1 - \frac{1}{3^n}$

So (x_n) is increasing

• $\{x_n\}$ is bounded by 1. so (x_n) converges by THM1.

In fact $1 = \text{L.U.B.}$ so the limit is 1.

③ $x_n = \frac{n^n}{3^{n^2}}$

Write $x_n = \frac{n^n}{3^{n^2}} = \frac{n^n}{(3^n)^n} = \left(\frac{n}{3^n}\right)^n \sim 0^\infty$.

Take $f(x) = \frac{x^x}{3^{x^2}} \sim \frac{\infty}{\infty}$ as $x \to \infty$ ⇝ We can use L'Hopital!

Take $\ln(f(x)) = x \ln x - x^2 \ln 3 = x^2\left(\frac{\ln x}{x} - \ln 3\right)$

Know $\lim\limits_{x\to\infty} \frac{\ln x}{x} \underset{\boxed{\text{L'Hop}}}{\overset{\frac{\infty}{\infty}}{=}} \lim\limits_{x\to\infty} \frac{1/x}{1} = \lim\limits_{x\to\infty} \frac{1}{x} = 0$.

So $\ln(f(x)) = x^2\left(\frac{\ln x}{x} - \underbrace{\ln 3}_{>1}\right) \longrightarrow -\infty$

Conclusion: $\lim\limits_{n\to\infty} x_n = e^{-\infty} = \boxed{0}$

Q: What can we say about $y_n = \frac{n^n}{3^{n^2}}$ for a ~~fixed~~?

If $a>0$, we have $3^{n^a} \xrightarrow[n\to\infty]{} \infty$ & we can use same ideas as before:

$$f(x) = \frac{x^x}{3^{x^a}} \sim \frac{\infty}{\infty} \qquad \ln(f(x)) = x\ln x - x^a \ln 3$$
$$= x^a\left(x\frac{\ln x}{x^a} - \ln 3\right)$$

• $\boxed{\text{If } a>1}$ ⇝ $\frac{x\ln x}{x^a} = \frac{\ln x}{x^{a-1}} \sim \frac{\infty}{\infty}$ by L'Hosp: $\cancel{\lim_{x\to\infty}\frac{\ln x}{x^{a-1}} = \lim_{x\to\infty}\frac{\frac{1}{x}}{(a-1)x^{a-2}}}$

$$\lim_{x\to\infty}\frac{\ln x}{x^{a-1}} = \lim_{x\to\infty}\frac{\frac{1}{x}}{(a-1)x^{a-2}} = \lim_{x\to\infty}\underbrace{\frac{1}{a-1}}_{>0}\frac{1}{x^{\overbrace{a-1}^{>0}}} = 0$$

So $$\lim_{x\to\infty}\ln(f(x)) = \lim_{x\to\infty}\underbrace{x^a}_{\infty}\Big(\underbrace{x\frac{\ln x}{x^a}}_{0} - \underbrace{\ln 3}_{>1}\Big) = -\infty$$

So $$\lim_{n\to\infty}\frac{n^n}{3^{n^a}} = e^{-\infty} = \boxed{0} \quad \text{if } \underline{a>1}$$

• $\boxed{\text{If } a=1}$: $y_n = \left(\frac{n}{3}\right)^n \sim \infty^\infty = 0$ so $\lim_{n\to\infty} y_n = \boxed{0}$.

• $\boxed{\text{If } 0<a<1}$ $\ln f(x) = x^a\left(x\frac{\ln x}{x^a} - \ln 3\right)$ but

$$\lim_{x\to\infty}\frac{\ln x}{x^{a-1}} = \lim_{x\to\infty} x^{\overbrace{1-a}^{>0}}\ln x = \infty$$

So $\ln f(x) = \underbrace{x^a}_{\infty}\Big(\underbrace{x\frac{\ln x}{x^a}}_{\infty} - \ln 3\Big) = \infty$ so $\lim_{n\to\infty} y_n = e^{\infty} = \boxed{\infty}$

• $\boxed{\text{If } a\le 0}$ write $y_n = \frac{n^n}{3^{n^{-b}}} = \frac{n^n}{3^{1/n^b}}$ for $b=-a>0$.

Now $\left.\begin{array}{l} 1/n^b \xrightarrow[n\to\infty]{} 0 \text{ so } 3^{1/n^b} \longrightarrow 3^0 = 1 \neq 0 \\ n^n \longrightarrow \infty \end{array}\right\}$ Limit Laws give $\lim_{n\to\infty} y_n = \infty$ if $a<0$

<u>Conclusion</u>: $$\lim_{n\to\infty}\frac{n^n}{3^{n^a}} = \begin{cases} 0 & a\ge 1 \\ \infty & a<1 \end{cases}$$