Lecture XLV : §13.3 Convergent & divergent series

§1 From sequences to series:

Def If $\{a_n\}_n = \{a_1, a_2, a_3, \dots\}$ is a sequence, the series with general term a_n is the expression $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots\dots$

Example (last week): $\sum_{n=0}^{\infty} \frac{1}{10^n} = \frac{1}{1} + \frac{1}{10} + \frac{1}{10^2} + \cdots$ = (geometric) series with general term $a_n = \frac{1}{10^n}$.

$= \frac{10}{9}$

- Attached to the series = sequence of partial sums

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$\vdots$$
$$S_n = a_1 + a_2 + \cdots + a_n = \sum_{j=1}^{n} a_j$$

Example above:

$$S_1 = 1$$
$$S_2 = 1 + \frac{1}{10}$$
$$\vdots$$
$$S_n = \underbrace{1 + \frac{1}{10} + \dots + \frac{1}{10^{n-1}}}_{n \text{ terms}}$$
$$\overset{(*)}{=} \frac{1 - 1/10^n}{1 - 1/10} = \frac{10}{9}\left(1 - \frac{1}{10^n}\right)$$

- Long Division (last week)

(*) $1 - x^n = (1-x)(1 + x + \dots + x^{n-1})$

Take $x = \frac{1}{10}$.

- Use the partial sums to define the sum of the series as a limit.

Def: $\sum_{j=1}^{\infty} a_j = \lim_{n\to\infty} \sum_{j=1}^{n} a_j = \lim_{n\to\infty} S_n$

Example above: $S_n = \frac{10}{9}\left(1 - \frac{1}{10^n}\right) \xrightarrow[n\to\infty]{} \frac{10}{9} \cdot 1 = \frac{10}{9}$ (by Limit Laws), where $\frac{1}{10^n} \xrightarrow{n\to\infty} 0$

so the series converges to $\frac{10}{9}$.

Def:
- If the limit of partial sums exists & it's finite ($= L$ in $\mathbb{R}$), we say that the series converges to L (or that L is the sum of the series)
- If the partial sums have no limit or limit is infinite ($\pm\infty$), we say that the series diverges.

Note. This is very similar to what we did with improper integrals!

$$\int_1^\infty f(x)\,dx = \lim_{t\to\infty}\int_1^t f(x)\,dx \rightsquigarrow \sum_{j=1}^\infty a_j = \lim_{n\to\infty}\sum_{j=1}^n a_j$$

- We will use the connection to Riemann Sums to compute various sums of series & show divergence (eg. harmonic series $\sum_{n=1}^\infty \frac{1}{n}$)

Example $\sum_{n=0}^\infty \left(\frac{1}{2}\right)^n$ has partial sums $S_n = \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} = 2\left(1-\frac{1}{2^{n+1}}\right) \xrightarrow[n\to\infty]{} 2(1-0) = 2$

So $\sum_{n=0}^\infty \frac{1}{2^n} = 2$.

§2 Geometric Series:

Recall: $1-x^{n+1} = \underbrace{(1-x)(1+x+\cdots+x^n)}_{\text{long division}} \rightsquigarrow 1+x+\cdots+x^n = \frac{1-x^{n+1}}{1-x}$ if $x\neq 1$

- ⚠ $\lim_{n\to\infty}\frac{1-x^{n+1}}{1-x} = \frac{1}{1-x}\left(1-\lim_{n\to\infty}x^{n+1}\right)$ only gives a number when $|x|<1$

So $\boxed{\sum_{n=0}^\infty x^n = \frac{1}{1-x} \quad \text{whenever } |x|<1}$ Geometric Series

Furthermore, the series is divergent if $|x|\geq 1$ because $\lim_{n\to\infty} x^{n+1} = \begin{cases}\infty & \text{if } x\geq 1\\ \text{oscillates} & \text{if } x\leq -1\end{cases}$

Variant: What if we start somewhere other than 0?

$$\boxed{\sum_{n=k}^\infty x^n} = x^k + x^{k+1} + x^{k+2} + \cdots = x^k\left(1+x+x^2+\cdots\right)$$
$$= x^k\sum_{j=1}^\infty x^j = \boxed{x^k\frac{1}{1-x} \quad \text{for } |x|<1}$$

If $k=0$, we recover the original formula

Example: $\sum_{n=3}^\infty \frac{1}{2^n} = \frac{1}{2^3}+\frac{1}{2^4}+\cdots = \frac{1}{8}\left(1+\frac{1}{2}+\cdots\right) = \frac{1}{8}2 = \frac{1}{4}$

Alternative: Add & substract missing terms

$$= 1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots - \left(1+\frac{1}{2}+\frac{1}{2^2}\right) = \sum_{n=0}^\infty\frac{1}{2^n} - \left(1+\frac{1}{2}+\frac{1}{4}\right)$$
$$= 2 - \frac{7}{4} = \frac{1}{4}$$

- Same alternative method works in general $= \sum_{n=0}^\infty x^n - \left(1+x+\cdots+x^{k-1}\right)$

$$= \frac{1}{1-x} - \frac{1-x^k}{1-x} = \frac{x^k}{1-x} \text{ for } |x|<1$$

Application: Decimal expansions with repeated patterns = rational numbers.

Why? • Say decimal expansion of z has repeating pattern with $0 \le x < 1$.

$$z = \underbrace{0.a_1 a_2 \cdots a_n}_{\text{non-repeating part}} \underbrace{\boxed{a_{n+1} \cdots a_{n+s}}}_{b_1 \;\cdots\; b_s} \underbrace{\boxed{a_{(n+s)+1} \cdots a_{n+2s}}}_{b_1 \;\cdots\; b_s} \cdots$$

s terms in repeating pattern

$$= \underbrace{\sum_{k=1}^{n} \frac{a_k}{10^k}}_{a \text{ in } \mathbb{Q}} + \underbrace{\frac{b_1}{10^{n+1}} + \cdots + \frac{b_s}{10^{n+s}}}_{y \text{ in } \mathbb{Q}} + \underbrace{\frac{b_1}{10^{n+1+s}} + \cdots + \frac{b_s}{10^{n+2s}}}_{\frac{1}{10^s} y \text{ in } \mathbb{Q}} + \cdots$$

$$= a + y + \frac{y}{10^s} + \frac{y}{10^{2s}} + \cdots$$

$$= a + y \underbrace{\left(1 + \frac{1}{10^s} + \frac{1}{(10^s)^2} + \cdots\right)}_{\text{geom series for } x = \frac{1}{10^s}}$$

$$= a + y \frac{1}{1 - \frac{1}{10^s}}$$

$$= a + y \frac{10^s}{10^s - 1} \qquad \text{so } \boxed{z \text{ in } \mathbb{Q}}$$

in $\mathbb{Q}$

• Now, pick a rational number ~~z in $(0,1)$~~ $z \ge 0$, write $z = \frac{p}{q} + k$ & do the long division. with $0<q<p$, k integer

Example:

$$\begin{array}{r} 3.142857 \\ 7 \,\overline{)\,22} \\ -21 \\ \hline 10 \\ -7 \\ \hline 30 \\ -28 \\ \hline 20 \\ -14 \\ \hline 60 \\ -56 \\ \hline 40 \\ -35 \\ \hline 50 \\ -49 \\ \hline 1 \end{array}$$

repeated remainder

$$\frac{22}{7} = 3.142857142857\ldots = 3.\overline{142857}$$

$$= \underset{a}{3} + \underbrace{\left(\frac{1}{10} + \frac{4}{10^2} + \frac{2}{10^3} + \frac{8}{10^4} + \frac{5}{10^5} + \frac{7}{10^6}\right)}_{= y} \cdot \left(1 + \frac{1}{10^6} + \frac{1}{10^{36}} + \cdots\right)$$

$$= a + y \frac{10^6}{10^6 - 1}.$$

General procedure: Long division says either we get a remainder 0 at some point (so repeated pattern = 0) or we repeat a remainder r. (always in $\{1, 2, \ldots, q-1\}$)