Lecture XLVI §13.3 Convergent & divergent series (II)
§13.4 General properties.

§1 Geometric series & applications

Recall Geometric series $= \sum_{i=0}^{\infty} x^i = 1 + x + x^2 + \cdots$ for x in $\mathbb{R}$

If $|x|<1$, $\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$, All other cases: series diverges

Motivation: $\frac{1}{1-x}$ is not a polynomial but we can write it as a series.

$\frac{1}{1-x} \stackrel{?}{=} 1 + x + x^2 + \cdots$ means $1 = (1-x)(1+x+x^2+\cdots)$

$\underset{\text{distribute}}{=} 1 + \cancel{x} + \cancel{x^2} + \cdots - \cancel{x} - \cancel{x^2} - \cdots$

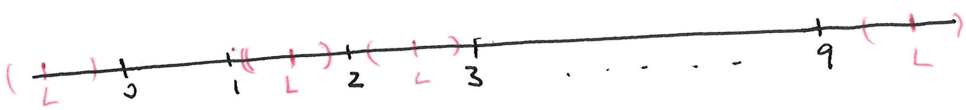
All terms cancel out!

We have a problem as soon as x goes from a symbolic variable to a numeric one precisely because we won't be able to cancel & $1 \stackrel{?}{=} \infty - \infty$ if $x > 1$.

Application: z is rational $\equiv$ decimal expansion of z has repeating pattern.

Application 2: The sequence $\{x_n\}_n = \{n^{th}$ decimal digit of $\pi\}$ diverges.

$x_n = 0, 1, \ldots, 9$. What can the limit L be?



CASE 1: L is not $0, 1, 2, \ldots, 9$

Take $\varepsilon = \min\{|0-L|, |1-L|, |2-L|, \ldots, |9-L|\} > 0$

Then $|0-L| \geq \varepsilon$, $|1-L| \geq \varepsilon$, $\ldots$, $|9-L| \geq \varepsilon$. so $|x_n - L| \geq \varepsilon$ for all n.

CASE 2 L is one of $0, 1, 2, \ldots, 9$.

Take $\varepsilon = 1$, if $\lim_{n\to\infty} x_n = L$, this means that we can find n_0 so that $|x_n - L| < 1$ for $n \geq n_0$. But this forces $x_n = L$ for $n \geq n_0$.

So π will have a decimal expansion with a repeating pattern!

We know this is not possible because π is NOT rational. **Conclusion:** there is NO limit so $\{x_n\}_n$ diverges.

Natural question: Manipulation of series? $\longleftrightarrow$ Limit Laws

① $\left.\begin{array}{l}\displaystyle\sum_{n=0}^{\infty} a_n = L \text{ in } \mathbb{R} \\ \lambda \text{ in } \mathbb{R}\end{array}\right\}$ $\rightsquigarrow$ $\underbrace{\sum_{n=0}^{\infty} (\lambda a_n)}_{\text{new series}} = \lim_{n\to\infty} \sum_{j=0}^{n} \lambda a_j = \lim_{n\to\infty} \lambda \sum_{j=0}^{n} a_j$

$$\underset{\substack{\uparrow\\ \text{Limit}\\ \text{Laws}\\ \text{for seq.}}}{=} \lambda \lim_{n\to\infty} \sum_{j=0}^{n} a_j = \lambda L.$$

② $\sum_{n=0}^{\infty} a_n = L$, $\sum_{n=0}^{\infty} b_n = T$. Then $\sum_{n=0}^{\infty} (a_n + b_n)$ converges & its sum is $L+T$

Why? $\sum_{n=0}^{\infty} (a_n+b_n) = \lim_{n\to\infty} \sum_{j=0}^{n} (a_j + b_j) = \lim_{n\to\infty} (a_1+b_1) + \cdots + (a_n+b_n)$

$$\underset{\text{rearrange finite sum}}{=} \lim_{n\to\infty} (a_1 + \cdots + a_n) + (b_1 + \cdots + b_n) = \lim_{n\to\infty} \sum_{j=0}^{n} a_j + \sum_{j=0}^{n} b_j$$

$$\underset{\substack{\uparrow\\ \text{Limit}\\ \text{Laws for seq}}}{=} \lim_{n\to\infty} \sum_{j=0}^{n} a_j + \sum_{m=0}^{\infty} \sum_{j=0}^{n} b_j = L + T.$$

⚠ We don't always have closed formulas for partial sums (like we do for geom. series). In these cases, want to decide convergence / diverge.

Other examples?

Ex 1: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots\cdots = 1.$

Why? general term $a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ (partial fraction decomposition of $\frac{1}{x(x+1)}$)

So $S_1 = a_1 = 1 - \frac{1}{2}$

$S_2 = a_1 + a_2 = \left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \tfrac{1}{3}\right) = 1 - \frac{1}{3}$

$\vdots$

$S_n = a_1 + a_2 + \cdots + a_n = \left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \cdots + \left(\cancel{\tfrac{1}{n}} - \tfrac{1}{n+1}\right) = 1 - \frac{1}{n+1}$

So $\lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{n+1} = 1$ $\rightsquigarrow$ sum of the series $= \boxed{1}$ ✓

General Telescopic series: $\sum_{n=1}^{\infty} a_n$ where $a_n = b(n) - b(n+1)$ for some function $b(x)$ on $\mathbb{R}_{>0}$.

Then $S_n = a_1 + a_2 + \cdots + a_n = \big(b(1) - b(2)\big) + \big(b(2) - b(3)\big) + \cdots + \big(b(n) - b(n+1)\big)$

$$= b(1) - b(n+1)$$

Prop Telescopic series converges if and only if $\lim_{n\to\infty} b(n) = L$ in $\mathbb{R}$

The sum $\sum_{n=1}^{\infty} \underbrace{(b(n) - b(n+1))}_{= a_n} = b(1) - L$ (Typically: $L = 0$)

§3 General properties

Prop 1: If $\sum_{n=0}^{\infty} a_n$ converges, then $a_n \xrightarrow[n\to\infty]{} 0$

Proof: Write $a_n = S_n - S_{n-1} \xrightarrow[n\to\infty]{} L - L = 0$ if $L = \sum_{n=0}^{\infty} a_n$.

Q: How to use this? If $a_n \not\xrightarrow[n\to\infty]{} 0$ we know for sure the series $\sum_{n=0}^{\infty} a_n$ diverges

Example $\sum_{n=0}^{\infty} (-1)^n$ diverges.

Q: What can we say if $a_n \geq 0$ for n large enough? (NAME: Series with positive terms)

Can we decide convergence?

Prop 2: Assume $\sum_{n=0}^{\infty} a_n$ is a series with positive terms. Then, the series converges if and only if the partial sums $\{S_n\}_n$ form a bounded sequence

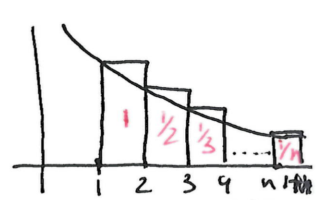
Proof: If $a_n \geq 0$ for $n \geq n_0$, then $S_{n+1} = S_n + a_{n+1} \geq S_n$ for $n \geq n_0$.

So partial sums are an increasing sequence. For these, we know that boundedness is the same as convergence. (THM 1 of Lecture 44).

Application: The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

$$S_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \geq \int_1^{n+1} \frac{1}{x}\,dx = \ln(n+1) \to \infty$$

So S_n is not bounded!



Alternative bounds? For every m integer, pick n satisfying $n > 2^{m+1}$

$$S_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} > S_{2^{m+1}} = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^{m+1}} \quad \text{(fewer terms)}$$

Now, we group by powers of 2

$$= \left(1+\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\cdots+\left(\frac{1}{2^m}+\frac{1}{2^m+1}+\cdots+\frac{1}{2^{m+1}}\right)$$

$$> \frac{1}{2}+\underbrace{\left(\frac{1}{4}+\frac{1}{4}\right)}_{2 \text{ terms}}+\underbrace{\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)}_{4 \text{ terms}}+\cdots+\underbrace{\left(\frac{1}{2^{m+1}}+\frac{1}{2^{m+1}}+\cdots+\frac{1}{2^{m+1}}\right)}_{2^m \text{ terms}}$$

$$= \frac{1}{2}+\frac{2}{4}+\frac{4}{8}+\cdots+\frac{2^m}{2^{m+1}}$$

$$= \underbrace{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots+\frac{1}{2}}_{m+1 \text{ summands}} = \frac{m+1}{2}$$

We conclude $S_n > \frac{m+1}{2}$ if $n > 2^{m+1}$

Since m is arbitrary, we see that the partial sums are <u>unbounded</u>, so $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges EVEN THOUGH the general term $\frac{1}{n}$ goes to 0!!

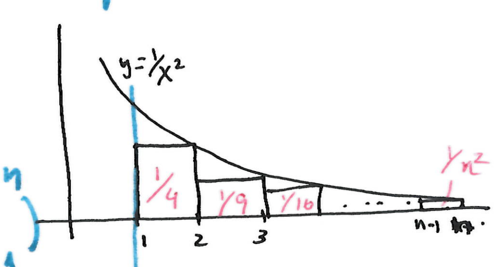
Q: At what rate of $a_n \xrightarrow[n\to\infty]{} 0$ do we get convergence of $\sum_{n=1}^{\infty} a_n$?

($\sum_{n=1}^{\infty} \frac{1}{n}$ diverges but $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges).

<u>Ex 2</u>: $a_n = \frac{1}{n^2}$ ⇝ positive series $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots = \frac{\pi^2}{6}$ (EULER)

To show convergence, we just need to bound the partial sums!

<u>Method 1</u>: $a_n = \frac{1}{n^2} = f(n)$ with $f(x) = \frac{1}{x^2}$



$$S_n = 1+\left(\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2}\right) \le 1+\int_1^n \frac{1}{x^2}\,dx = 1+\left(\frac{-1}{x}\Big|_1^n\right)$$

$$= 1+\left(\frac{-1}{n}+1\right) = 2-\frac{1}{n} < 2 \text{ for } \underline{\text{all}}\ n$$

So we conclude $S_n < 2$ for <u>all</u> n

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ has positive terms $\Big\}$, so it converges (Prop 2)!

Also get sum ≤ 2.

Method 2: Work out a bound as with the harmonic series.

$$S_n = 1 + \frac{1}{2\cdot 2} + \frac{1}{3\cdot 3} + \cdots + \frac{1}{n\cdot n} < 1 + \underbrace{\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(n-1)n}}_{\text{(Looks like Telescopic Example!}} =$$

$1<2$
$2<3$
$\vdots$

$$= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) = 1 + \left(1 - \frac{1}{n}\right) = 2 - \frac{1}{n}$$

Conclusion: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, but the sum of some of its terms $\left(= \frac{1}{n^2}\right)$ converges!

Ex 3: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges (To $\ln 2$).