Lecture XLVII : §13.4 General properties (II)
§13.6 The integral test.

§1 More examples:

Ex 1: $\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = e$ $\quad (0! = 1! = 1$ by def$)$

Why is it convergent? Series with positive terms, so only need to check partial sums are a bounded sequence.

$s_0 = 1$

$s_1 = 1+1$

$s_2 = 1+1+\frac{1}{2!}$

$s_3 = 1+1+\frac{1}{2!}+\frac{1}{3!}$

$\rightsquigarrow$ $s_n = 1 + 1 + \frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \cdots + \frac{1}{2\cdot 3\cdots(n)}$ for $n \geq 2$

$\leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}}$

$= 1 + \sum_{m=0}^{n-1} \frac{1}{2^m} < 1 + \sum_{m=0}^{\infty} \frac{1}{2^m} = 1 + 2 = 3$

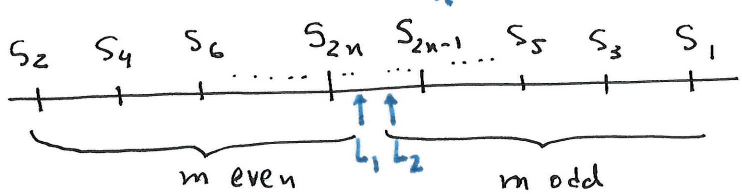
So $s_n < 3$ for all $n \geq 2$. So the series converges & the sum < 3.

[We'll see $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$ for all x, so taking $x = 1$ gives the sum $= e$]

Application: Can show e is irrational (future lecture)

Ex 2: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2.$

Main strategy: $|a_n| = \frac{1}{n} \longrightarrow 0$ & a_n alternates signs.



$S_1 = 1 > 0$

$S_2 < S_1$

$S_1 > S_3 > S_2$

$S_2 < S_4 < S_3$

$S_1 = 1$

$S_2 = 1 - \frac{1}{2} = \frac{1}{2} < S_1$

$S_3 = S_2 + \frac{1}{3}$

$= S_1 + \underbrace{\frac{1}{3} - \frac{1}{2}}_{<0}$

$S_4 = S_3 - \frac{1}{4}$

$= S_2 + \underbrace{\frac{1}{3} - \frac{1}{4}}_{>0}$

We get

$$\boxed{\begin{array}{l} S_{2(n-1)} < S_{2n} < S_{2n-1} \\ S_{2(n-1)} < S_{2n-1} < S_{2n-3} \end{array}}$$

. Treat even & odd partial sums separately:

- $(S_{2k})_{k \in \mathbb{N}}$ increases & its bounded (by S_1), so it converges to L_1
- $(S_{2k+1})_{k \in \mathbb{N}}$ decreases ———— (by S_2), so ———— L_2

Can show $\boxed{L_1 = L_2}$ because $S_{2k} - S_{2k-1} = \frac{(-1)^{2k+1}}{2k} = \frac{-1}{2k} \xrightarrow[k \to \infty]{} 0$

Q: What is the effect of rearranging sums?

Example $\sum_{n=0}^{\infty}(-1)^n = 1-1+1-1+\cdots$ diverges $\quad (-1)^n \not\to 0$ as $n\to\infty$

but $\sum_{n=1}^{\infty}\left((-1)^{2n}-(-1)^{2n-1}\right) = (1-1)+(1-1)+\cdots = 0$ converges

Conclusion: Rearranging the sum by inserting parenthesis as above can change the value of the series if the series diverges. However such rearrangement will preserve the sum if the original series converges.

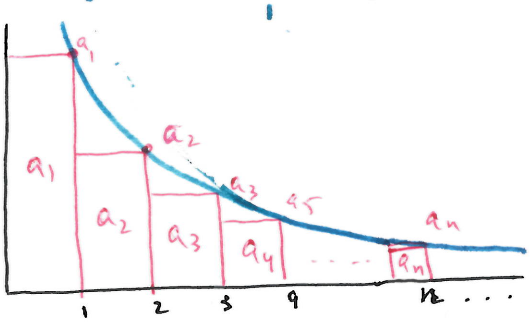
- For series with positive terms, this will NEVER happen (Appendix A13)

§2 The Integral Test. Euler's Constant

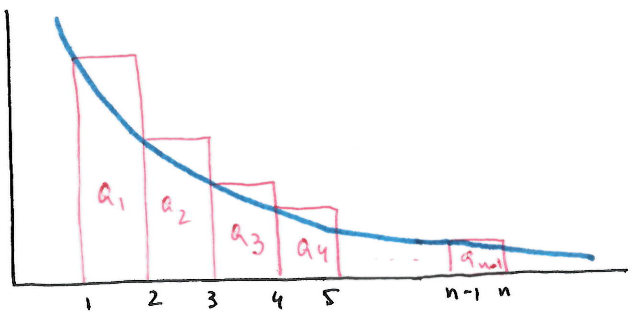
GOAL: Use improper integrals to test for convergence of $\sum_{n=1}^{\infty} a_n$

- What functions do we use?
  - Need $f:(0,\infty)\longrightarrow\mathbb{R}$ with $f(n)=a_n$ & f continuous
  - Need $a_n \geq 0$ for all n & $(a_n)_n$ decreasing $\rightsquigarrow$ $f_{(x)}\geq 0$ & $f_{(x)}$ decreasing ($f'\leq 0$ for example)

IDEA: Relate $\int_1^{\infty} f_{(x)}\,dx$ to lower & Upper Riemann Sums



LOWER SUMS



UPPER SUMS

f decreasing & $f_{(x)}\geq 0$ gives:

$$a_2+a_3+\cdots+a_n \leq \int_1^n f_{(x)}\,dx \leq a_1+\cdots+a_{n-1}$$

So
$$\boxed{\sum_{j=1}^{n} a_j} \leq a_1 + \int_1^n f_{(x)}\,dx \leq a_1 + \int_1^{n+1} f_{(x)}\,dx \leq a_1 + \boxed{\sum_{j=1}^{n} a_j}$$

(Add a_1; $\int_n^{n+1} f_{(x)}\,dx \geq 0$)

Cauchy Integral Test: If $f:[1,\infty)\to\mathbb{R}_{\geq 0}$ decreasing & $a_n=f(n)$ for all n, then either $\sum_{n=1}^{\infty} a_n$ & $\int_1^{\infty} f_{(x)}\,dx$ both converge or both diverge.

Proof : Comparing lower & upper Riemann sums, we get:

$$\boxed{\sum_{j=2}^{n} a_j} \leq a_1 + \boxed{\int_1^n f(x)\,dx} \leq a_1 + \boxed{\sum_{j=1}^{n} a_j}$$

Note: (1) $S_n = \sum_{j=1}^{n} a_j$ is increasing because $a_n \geq 0$ for all n.

(2) $\lim_{n\to\infty} \int_1^n f(x)\,dx = \int_1^\infty f(x)\,dx$. ; $\boxed{b_n := \int_1^n f(x)\,dx}$ is increasing with n

The inequalities above will show that S_n & $\int_1^n f(x)\,dx$ are both bounded above or both unbounded.

This means b_n & S_n both converge or diverge.

Formally: $S_n \leq a_1 + b_n \leq a_1 + \int_1^\infty f(x)\,dx$ & $b_n \leq S_n$

- So if $\int_1^\infty f(x)\,dx$ converges, then S_n is bounded by $1 + \int_1^\infty f(x)\,dx$ & so S_n also converges.
- Similarly, if $\int_1^\infty f(x)\,dx$ diverges then $\int_1^n f(x)\,dx \leq S_n$ (with $\int_1^n f(x)\,dx \to \infty$) forces S_n to diverge.

<u>Conclusion</u>: The series $\sum_{n=1}^{\infty} f(n)$ & the improper integral $\int_1^\infty f(x)\,dx$ have the <u>SAME</u> behavior.

On concrete examples, we'll know how they behave (converge or diverge)