Lecture XLVIII: §13.6 The integral test. Euler's Constant
§13.7 The Ratio Test & the Root Test

§1 Integral Test

$f:[1,\infty)\longrightarrow \mathbb{R}_{\geq 0}$ continuous, decreasing & $f(n)=:a_n$.

Compare Lower & Upper R.S with area under the curve to get

$$\boxed{\underbrace{a_1+\cdots+a_n}_{=:S_n}} \leq a_1+\boxed{\underbrace{\int_1^n f(x)\,dx}_{=:b_n}} \leq a_1+\boxed{\underbrace{(a_1+\cdots+a_n)}_{=:S_n}} \qquad (*)$$

Test: S_n converges if and only if $b_n:=\int_1^n f(x)\,dx$ converges.

Examples ① $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges because $\int_1^n \frac{1}{x}\,dx = \ln(x)\Big|_1^n=\ln n$ diverges & $f(x)=1/x$ cont & decr.

② $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges " $\int_1^n \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_1^n = 1-\frac{1}{n}$ converges & $f(x)=\frac{1}{x^2}$ cont & decr.

③ General p-series $=\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges if and only if $p>1$.

Why? If $p<0$ $\frac{1}{n^p}=n^{-p}$, $-p>0$, $\xrightarrow[n\to\infty]{}\infty$ If $p=0$ $\frac{1}{n^p}=\frac{1}{n^0}=1 \underset{n\to\infty}{\not\to} 0$

so diverges as well. Thus, we ~~mainly~~ need to focus on $\boxed{p>0}$ & $\boxed{p\neq 1}$ (Ex①)

Take $f(x)=\frac{1}{x^p}$ on $[1,\infty)$ cont & diff'ble. & $f(x)\geq 0$ on $[1,\infty)$

$f'(x)=\frac{-p}{x^{p+1}}<0$ on $[1,\infty)$ so f is decreasing.

$$\int_1^n f(x)\,dx = \int_1^n \frac{dx}{x^p} \underset{p\neq 1}{=} \frac{x^{-p+1}}{-p+1}\Bigg|_1^n = \frac{1}{1-p}\left(n^{1-p}-1\right)$$

So $\int_1^{\infty} f(x)\,dx = \lim_{n\to\infty}\frac{1}{1-p}(n^{1-p}-1) = \frac{1}{1-p}\lim_{n\to\infty}(n^{1-p}-1) = \begin{cases}\infty & \text{if } 1-p>0\\ \frac{-1}{1-p}=\frac{1}{p-1} & \text{if } 1-p<0\end{cases}$

So $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges if and only $p>1$ (by Integral Test

④ $\sum_{n=2}^{\infty} \frac{1}{n \ln n} = ?$ ⇝ $f(x) = \frac{1}{x \ln x}$ defined on $[2, \infty)$ & continuous, positive.

$$f'(x) = \frac{-1}{(x \ln x)^2} \underbrace{(\ln x + 1)}_{> 0 \text{ for } x > 2} < 0 \quad ⇝ \quad f \text{ is decreasing on } [2, \infty)$$

Test say $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converges if and only if $\int_2^{\infty} \frac{dx}{x \ln x}$ does

$$\int_2^n \frac{dx}{x \ln x} \underset{\substack{u = \ln x \\ du = \frac{dx}{x}}}{=} \int_{\ln 2}^{\ln n} \frac{du}{u} = \ln u \Big|_{\ln 2}^{\ln n} = \underbrace{\ln(\underset{\downarrow n\to\infty}{\ln n}) - \ln(\ln 2)}_{\downarrow n \to \infty \; \infty} \xrightarrow[n \to \infty]{} \infty$$

Conclusion: The series diverges

⑤ General: $\sum_{n=2}^{\infty} \frac{1}{n (\ln n)^p} = ??$ ⇝ $f(x) = \frac{1}{x (\ln x)^p}$ on $[2, \infty)$ cont, positive

$$f'(x) = \frac{-1}{(x (\ln x)^p)^2} \left((\ln x)^p + p (\ln x)^{p-1} \right) = - \frac{\overbrace{(\ln x)^{p-1}}^{> 0}}{\underbrace{x^2 (\ln x)^{2p}}_{> 0}} \underbrace{(p + \ln x)}_{> 0} < 0$$

so f is decreasing on $[2, \infty)$

- Note: $(\ln n)^p \le \ln n$ for $p \le 1$ & $n \ge 3$. ($\ln n \ge 0$)

so $\frac{1}{n (\ln p)^p} \ge \frac{1}{n \ln n}$ for $p \le 1$ & so $\sum_{n=2}^{\infty} \frac{1}{n (\ln n)^p} \ge \underbrace{\sum_{n=2}^{\infty} \frac{1}{n \ln n}}_{\text{diverges by ④}}$

so $\boxed{\sum_{n=2}^{\infty} \frac{1}{n (\ln n)^p} \text{ also diverges for } p \le 1}$ (Example of Comparison Thms).

- For $p > 1$, we use Integral Test

$$\int_2^n \frac{dx}{x (\ln x)^p} \underset{\substack{u = \ln x \\ du = \frac{dx}{x}}}{=} \int_{\ln 2}^{\ln n} \frac{du}{u^p} \underset{p \ne 1}{=} \frac{u^{-p+1}}{-p+1} \Big|_{\ln 2}^{\ln n} = \frac{1}{1-p} \left((\ln n)^{\overset{< 0}{1-p}} - (\ln 2)^{1-p} \right)$$

$$= \frac{1}{1-p} \left(\boxed{\frac{1}{(\ln n)^{p-1}}} \xrightarrow{n \to \infty} 0 \; - \frac{1}{(\ln 2)^{p-1}} \right)$$

So $\boxed{\sum_{n=2}^{\infty} \frac{1}{n (\ln n)^p} \text{ converges for } p > 1}$ & sum $\le \frac{1}{(p-1)(\ln 2)^{p-1}} + \frac{1}{2 \ln 2} = \int_2^{\infty} f(x)\,dx + a_2$

Q What else can we learn from the ineq. in (*)?

$(*) - \int_1^n f(x)\,dx$ gives: $\quad 0 \leq \boxed{\sum_{j=1}^{n} a_j - \int_1^n f(x)\,dx} \leq a_1$

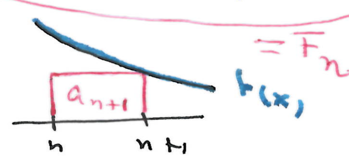
(RHS) of (*) $\quad =: F_n \quad$ (LHS) of (*)

We have: • $\{F_n\}_n$ is bounded $(0 \leq F_n \leq a_1$.

• $\{F_n\}$ is decreasing $\quad F_{n+1} = F_n + a_{n+1} - \int_n^{n+1} f(x)\,dx \leq F_n$

(because $F_{n+1} = S_{n+1} - \int_1^{n+1} f(x)\,dx = \underbrace{S_n + a_{n+1} - \int_1^n f(x)\,dx}_{=F_n} - \int_n^{n+1} f(x)\,dx)$

And $a_{n+1} - \int_n^{n+1} f(x)\,dx \leq 0$ because



Conclusion $\{F_n\}_n$ is decreasing & bounded, so it's convergent. Say $L = \lim_{n\to\infty} F_n$

We have $\boxed{0 \leq L = \lim_{n\to\infty} \left(a_1 + \cdots + a_n - \int_1^n f(x)\,dx\right) \leq a_1}$

Application: Take $a_n = \frac{1}{n}$ & $f(x) = \frac{1}{x}$ $\rightsquigarrow$ $\int_1^n f(x)\,dx = \ln n$.

So $\lim_{n\to\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n\right) = L$ & $0 \leq L \leq a_1$.

or $\lim_{n\to\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - (\ln n + L)\right) = 0$

Name: $\boxed{L = \gamma = \text{"Euler's Constant"}}$ $\quad (\gamma = 0.5772156649\ 0153286060 \ldots$

Open Q: Is γ in $\mathbb{Q}$ or not?)

Algorithmic notation: $b_n = o(1)$ if $\frac{b_n}{1} \longrightarrow 0$ so write lim above

as $\boxed{1 + \frac{1}{2} + \cdots + \frac{1}{n} = \ln n + \gamma + o(1).}$

Consequence: $\quad 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2.$

• Treat even & odd partial sums separately to confirm convergence (last time)

• Now, look closer at the partial sums S_{2m} $\left(\lim_{n\to\infty} S_{2m+1} = \lim_{n\to\infty} S_{2m} = \text{sum of series}\right)$

$$S_{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}$$

$$\underset{\text{separate}}{\overset{+\,\&\,-}{=}} \left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}\right) - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n}\right)$$

ODD denominators EVEN denominators

add & substr even denom

$$= \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n-1} + \frac{1}{2n}\right) - 2\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n}\right)$$

clear even den.

$$= 1 + \frac{1}{2} + \cdots + \frac{1}{2n} - \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)$$

$$= \ln 2n + \gamma + o(1) - (\ln n + \gamma + o(1))$$

$$= \ln 2n - \ln n + o(1) = \ln \frac{2n}{n} + o(1) = \ln 2 + o(1)$$

So $S_{2n} = \ln(2) + o(1) \xrightarrow[n \to \infty]{} \boxed{\ln 2}$

Conclusion: The sum of the series is $\ln 2$.

§2 The ratio Test

Motivation: $\displaystyle\sum_{n=0}^{\infty} r^n = \begin{cases} \text{converges } \left(\text{to } \frac{1}{1-r}\right) & \text{if } 0 \leq \boxed{r < 1} \\ \text{diverges} & \text{if } \boxed{r > 1} \end{cases}$ for $r \geq 0$

For the geometric series, the ratio between successive terms is the constant value r: $\dfrac{a_{n+1}}{a_n} = \dfrac{r^{n+1}}{r^n} = r$.

So if $r < 1$ $a_{n+1} = r\, a_n$ & the terms decay sufficiently fast to ensure the series converges.

The ratio test says this always works for series of positive terms

Ratio Test Pick a sequence $\{a_n\}_n$ with $\boxed{a_n > 0}$ for n large enough.

Assume $\displaystyle\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = L$ exists. Then:

$$\sum_{n=1}^{\infty} a_n = \begin{cases} \text{converges} & \text{if } L < 1 \\ \text{diverges} & \text{if } L > 1 \\ \text{don't know} & \text{if } L = 1 \to \text{ANYTHING CAN HAPPEN} \end{cases}$$

Examples: ① $\sum_{k=0}^{\infty} \frac{1}{k!}$ converges (to e)

Proof 1: Careful bounding techniques (Lecture 47)

Proof 2: Use Ratio Test $a_n = \frac{1}{n!} > 0$, $\frac{a_{n+1}}{a_n} = \frac{n!}{(n+1)!} = \frac{1}{n+1} \xrightarrow[n\to\infty]{} 0 < 1$

so convergence follows!

② $\sum_{k=0}^{\infty} \frac{3^k}{k!}$ converges (to e^3)

Ratio Test: $a_n = \frac{3^n}{n!} > 0$, $\frac{a_{n+1}}{a_n} = \frac{3^{n+1}}{(n+1)!} \frac{n!}{3} = \frac{3}{n+1} \xrightarrow[n\to\infty]{} 0 < 1$.

③ $\sum_{k=0}^{\infty} \frac{k^6}{3^k}$ converges

Ratio Test $a_n = \frac{n^6}{3^n} > 0$, $\frac{a_{n+1}}{a_n} = \frac{(n+1)^6}{3^{n+1}} \cdot \frac{3^n}{n^6} = \left(\frac{n+1}{n}\right)^6 \frac{1}{3} \xrightarrow[n\to\infty]{} \frac{1}{3} < 1$

Remark: Ratio Test is useful if series involves factorials, powers, exponentials, products in general

Proof: Future Lecture

§3 The Root Test:

Motivation from geometric series. Extremely useful for power series (eg Taylor series)

Root Test: Pick a sequence $\{a_n\}$ with $\boxed{a_n > 0}$ for n large enough.

Assume $\lim_{n\to\infty} \sqrt[n]{a_n} = L$ exists. Then:

$$\sum_{n=1}^{\infty} a_n = \begin{cases} \text{converges} & \text{if } L < 1 \\ \text{diverges} & \text{if } L > 1 \\ \text{don't know} & \text{if } L = 1 \end{cases} \to \text{see examples ① \& ② below.}$$

Examples: ① $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges & $\sqrt[k]{\frac{1}{k}} = \frac{1}{k^{1/k}} \xrightarrow[n\to\infty]{} 1 = e^0$ (because $\ln(k^{1/k}) = \frac{\ln k}{k} \to 0$)

② $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges & $\lim_{k\to\infty} \sqrt[k]{\frac{1}{k^2}} = \left(\lim_{k} \sqrt[k]{\frac{1}{k}}\right)^2 = 1^2$

③ $\sum_{n=1}^{\infty} \frac{1}{n^n}$ converges by Root Test $\sqrt[n]{\frac{1}{n^n}} = \frac{1}{n} \xrightarrow[n\to\infty]{} 0 < 1$.

④ $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}$ converges by Root Test $\displaystyle\sqrt[n]{\frac{1}{(\ln n)^n}} = \frac{1}{\ln n} \xrightarrow[n\to\infty]{} 0 < 1$

⑤ $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ ———————— $\displaystyle\sqrt[n]{\frac{n^2}{2^n}} = \frac{\left(\sqrt[n]{n}\right)^2}{2} \xrightarrow[n\to\infty]{} \frac{1}{2} < 1$

⑥ $\displaystyle\sum_{n=1}^{\infty} \left(\sqrt[n]{n}-1\right)^{n/2}$ ———————— $\displaystyle\sqrt[n]{\left(\sqrt[n]{n}-1\right)^{n/2}} = \left(\sqrt[n]{n}-1\right)^{1/2} \xrightarrow[n\to\infty]{} 0 < 1$

Remark: Root Test is useful for series involving exponentials & powers.

Proof: Future Lecture