# Lecture XLVII: §13.5 Series of Non-negative terms. Comparison tests

**Comparison Test 1**: Assume $(a_n)$, $(b_n)$ are two sequences of non-negative terms

Assume $0 \leq a_n \leq b_n$ for all $n$ (for all $n \geq n_0$)

Then (1) If $\sum_{n=1}^{\infty} b_n$ converges to $L$, then $\sum_{n=1}^{\infty} a_n$ converges to $L' \leq L$

$(L' \leq L + (a_1 - b_1) + \cdots + (a_{n_0-1} - b_{n_0-1}))$

(2) If $\sum_{n=1}^{\infty} a_n$ diverges, so does $\sum_{n=1}^{\infty} b_n$.

**Example**: (2) $\ln(n) < n$ for $n \geq 1$ (since $y = \ln(x)$ lies below the line $y = x$)

so $\frac{1}{\ln(n)} > \frac{1}{n} > 0$

Since $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, so does $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$.

(1) $\sum_{n=1}^{\infty} \frac{1}{3^n+1}$ converges because $\sum_{n=1}^{\infty} \frac{1}{3^n}$ does & $L' \leq \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$.

**Proof**: Assume $0 \leq a_n \leq b_n$ for all $n$. & take partial sums on both sides

$$0 \leq \underbrace{a_1 + \cdots + a_n}_{= S_n} \leq \underbrace{b_1 + \cdots + b_n}_{t_n}$$

- $S_n$ & $t_n$ are both increasing

So if $t_n \xrightarrow[n\to\infty]{} L$ we get $(S_n) \leq L$ for all $n$ $\Big\}$ $\lim_{n\to\infty} S_n = \sum_{n=1}^{\infty} a_n$ exists & $\leq L$.

If $S_n \xrightarrow[n\to\infty]{} \infty$ (because $S_n$ is increasing & divergent, it can't be bounded)

then $t_n$ is also not bounded & so $t_n \xrightarrow[n\to\infty]{} \infty$. □

- We can generalize the result if $0 \leq a_n \leq b_n$ for all $n \geq n_0$ by rewriting both series by removing the first $n_0$ terms in both series:

$$\sum_{n=1}^{\infty} a_n = a_1 + \cdots + a_{n_0} + \boxed{\sum_{n=n_0+1}^{\infty} a_n}$$

$$\sum_{n=1}^{\infty} b_n = b_1 + \cdots + b_{n_0} + \boxed{\sum_{n=n_0+1}^{\infty} b_n}$$

→ comparison test holds for the tail, so it holds for both series.

**Example**: $\sum_{n=1}^{\infty} \frac{n+1}{n^n}$ converges by comparing it to $\frac{2}{n^2}$.

$$\frac{n+1}{n^n} = \frac{1}{n^{n-1}} + \frac{1}{n^n} < \frac{2}{n^{n-1}} \leq \frac{2}{n^2} \text{ for } n \geq 3$$

- For $n \geq 3$ $n-1 \geq 2$ & $n^{n-1} \geq n^2$ so $\frac{1}{n^{n-1}} \leq \frac{1}{n^2}$
- $\sum^{\infty} \frac{1}{n^2}$ converges, so does $\sum^{\infty} \frac{2}{n^2}$

**Limit Comparison Test**: Suppose $(a_n)$, $(b_n)$ non-negative terms, $b_n > 0$ for all $n$. Assume $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = L \neq 0$ (so $L>0$). Then either both series $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ converge or they both diverge.

**Why?** If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = L > 0$ then for $\varepsilon = \frac{L}{2}$ we can find $n_0$ for which

$$\frac{a_n}{b_n} \in (L-\varepsilon, L+\varepsilon) = \left(\frac{L}{2}, \frac{3L}{2}\right) \quad \text{so } \frac{L}{2} < \frac{a_n}{b_n} < \frac{3L}{2}$$

Since $b_n > 0$ we get $\quad \frac{L}{2} b_n < a_n < \frac{3L}{2} b_n =: d_n$

. Since multiplying by $\underset{0\neq}{\frac{L}{2}}$ & $\underset{\neq 0}{\frac{3L}{2}}$ does not affect the convergence we conclude

- If $\sum_{n=1}^{\infty} b_n$ converges, so does $\sum_{n=1}^{\infty} \frac{3L}{2} b_n$. By comparison test for $0 \le a_n < d_n$ we conclude $\sum_{n=1}^{\infty} a_n$ converges.
- If $\sum_{n=1}^{\infty} a_n$ converges, by comparison test for $0 \le \frac{L}{2} b_n \le a_n$, so does $\sum_{n=1}^{\infty} \frac{L}{2} b_n = \frac{L}{2} \sum_{n=1}^{\infty} b_n$, Hence, so does $\sum_{n=1}^{\infty} b_n$.
- If $\sum_{n=1}^{\infty} b_n$ diverges, so does $\sum_{n=1}^{\infty} \frac{L}{2} b_n$; By comparison test, so does $\sum_{n=1}^{\infty} a_n$.
- If $\sum_{n=1}^{\infty} a_n$ diverges, so does $\sum_{n=1}^{\infty} \frac{3L}{2} b_n$ (by comparison test). Hence, so does $\sum_{n=1}^{\infty} b_n$.

**Example**: $\sum_{n=1}^{\infty} \underbrace{\frac{n+2}{2n^3-3}}_{=:a_n}$ $\quad a_n \ge 0$. $\quad a_n = \frac{1+2/n^2}{2n^2 - 3/n}$ we compare it to $0 < \frac{1}{n^2} = b_n$ which converges.

Since
$$\frac{a_n}{b_n} = \frac{\frac{1+2/n^2}{2n^2-3/n}}{1/n^2} = n^2 \frac{1+\frac{2}{n^2}}{2n^2-\frac{3}{n}} = \frac{1+2/n^2}{2-\frac{3}{n^3}} \xrightarrow[n\to\infty]{} \frac{1}{2} > 0$$

**Remark**: Limit Comparison Test are useful if we compare with $\underset{(\text{DIV})}{\sum_{n=1}^{\infty} \frac{1}{n}}$, $\sum_{n=1}^{\infty} \frac{1}{n^2}$, $\sum_{n=1}^{\infty} \frac{1}{n!}$, $\sum_{n=1}^{\infty} x^n$ (conv. if and only if $|x|<1$)

**Example 2** (p-series) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p>1$ & diverges if $p \le 1$.

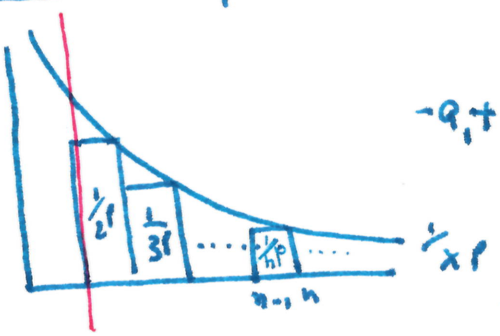
**Why?** If $p \le 1$, then $n^p \le n$ so $\underset{\text{DIV}}{\frac{1}{n}} \le \frac{1}{n^p}$ by Comparison Test, $\sum \frac{1}{n^p}$ diverges for $p \le 1$

If $p > 1$, we'll show that the partial sums sequence $(s_n)_n$ is str. increasing and bounded.

- If $p \geq 2$ we know it converges by comparison with the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ because $\frac{1}{n^p} \leq \frac{1}{n^2}$ $\forall p \geq 2$.

- Missing cases $1 < p < 2$.

Method 1: Compare to area under the curve $f(x) = \frac{1}{x^p}$: [Cauchy Integral Test (preview)]




$$-a_1 + s_n = \sum_{j=2}^{n} \frac{1}{j^p} < \int_1^{\infty} \frac{1}{x^p}\,dx = \left.\frac{x^{1-p}}{1-p}\right|_1^{\infty} = \frac{-1}{1-p} = \frac{1}{p-1} < \infty$$

So $s_n \leq \frac{1}{p-1} + a_1$ for all $n$,

so it converges (because $s_{n+1} = s_n + \underbrace{a_{n+1}}_{>0} > s_n$ for all $n$)

Method 2 [Similar to bounding techniques we saw for $\sum_{n=1}^{\infty} \frac{1}{n}$ & $\sum_{n=1}^{\infty} \frac{1}{n!}$ series]

- Fix $n$ & show $s_n < \frac{2^{p-1}}{2^{p-1}-1}$.

How? Pick $m$ with $n < 2^m$. Then

$$s_n = \sum_{k=1}^{n} \frac{1}{k^p} \leq \sum_{k=1}^{2^m-1} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots + \frac{1}{(2^m-1)^p}$$

(more terms!)

group by powers of 2

$$= 1 + \underbrace{\left(\frac{1}{2^p} + \frac{1}{3^p}\right)}_{2 \text{ terms}} + \underbrace{\left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right)}_{4 \text{ terms}} + \dots + \underbrace{\left(\frac{1}{(2^{m-1})^p} + \dots + \frac{1}{(2^m-1)^p}\right)}_{2^{m-1} \text{ terms}}$$

largest term   largest term   largest term

$$< 1 + \frac{2}{2^p} + \frac{4}{4^p} + \dots + \frac{2^{m-1}}{(2^{m-1})^p}$$

$$= 1 + \frac{1}{2^{p-1}} + \frac{1}{(2^2)^{p-1}} + \dots + \frac{1}{(2^{m-1})^{p-1}}$$

Write $a = \frac{1}{2^{p-1}}$:

$$= 1 + a + a^2 + \dots + a^{m-1} \leq \sum_{n=1}^{\infty} a^n \overset{\substack{|a|<1 \text{ because} \\ p>1}}{=} \frac{1}{1-a} = \frac{2^{p-1}}{2^{p-1}-1}$$

Conclusion: $s_n < \frac{2^{p-1}}{2^{p-1}-1}$ for all $n$, so $(s_n)$ is increasing & bounded so it must converge & $\sum_{n=1}^{\infty} \frac{1}{n^p} \leq \frac{2^{p-1}}{2^{p-1}-1}$.