Lecture L: §13.5 Comparison Tests
§13.7 Ratio & Root Tests

Comparison Test 1: Assume $\{a_n\}_n$, $\{b_n\}_n$ are sequences of non-negative terms ($a_n, b_n \geq 0$ for n large enough).

Assume $0 \leq a_n \leq b_n$ for n large enough. Then:

(1) If $\sum_{n=1}^{\infty} b_n = L$, then $\sum_{n=1}^{\infty} a_n$ converges with sum $\leq L$.

(2) If $\sum_{n=1}^{\infty} a_n$ diverges, so does $\sum_{n=1}^{\infty} b_n$.

Why? Compare sequences of partial sums (both increasing).

Typical examples to compare with: $\underset{(\text{div})}{\frac{1}{n}}$, $\underset{(\text{conv})}{\frac{1}{n^2}}$.

Limit Comparison Test: Assume $\{a_n\}$, $\{b_n\}$ are sequences with non-negative terms & $b_n > 0$ for all n large enough. Assume $\lim_{n \to \infty} \frac{a_n}{b_n} = L \neq 0$ (so $L > 0$)

Then $\sum_{n=1}^{\infty} a_n$ & $\sum_{n=1}^{\infty}$ have the same behavior: either both converge or both diverge.

Why? ~~The~~ limit definition gives $\frac{L}{2} b_n \leq a_n \leq \frac{3L}{2} b_n$ for n large enough $\leadsto$ Comparison Test 1 + limit laws give the statement.

Typical examples to compare with $b_n = \underset{(\text{DIV})}{\frac{1}{n}}$, $\underbrace{\frac{1}{n^2}, \frac{1}{n!}}_{(\text{CONV})}$, x^n ↳ conv/div depends on $|x|<1$ or not.

Application 1: Proofs of Ratio & Root Test

Ratio & Root Tests: Pick $\{a_n\}$ sequence of positive terms (for n large enough)

Assume $\underset{(\text{RATIO TEST})}{\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = L}$ or $\underset{(\text{ROOT TEST})}{\lim_{n \to \infty} \sqrt[n]{a_n} = M}$ exist. Then:

$$\sum_{n=1}^{\infty} a_n = \begin{cases} \text{converges} & \text{if } L<1 \;\Big|\; M<1 \\ \text{diverges} & \text{"} \quad L>1 \;\Big|\; M>1 \\ \text{don't know} & \text{"} \quad L=1 \;\Big|\; M=1 \end{cases}$$

$\leftarrow$ ANYTHING CAN HAPPEN

RATIO TEST   ROOT TEST

Appendix 12 has a refinement if limit 1 comes from below vs above.

Examples ① $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges & $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges (Ratio & Root Test with $L = M = 1$)

• $\frac{\frac{1}{k+1}}{\frac{1}{k}} = \frac{k}{k+1} \xrightarrow[k\to\infty]{} 1$, $\boxed{\sqrt[k]{k} \xrightarrow[n\to\infty]{} 1}$ because $\ln \sqrt[k]{k} = \frac{\ln k}{k} \xrightarrow[k\to\infty]{} 0$ by L'Hopital.

• $\frac{\frac{1}{(k+1)^2}}{\frac{1}{k^2}} = \left(\frac{k}{k+1}\right)^2 \xrightarrow[k\to\infty]{} 1^2 = 1$, $\sqrt[k]{k^2} = \left(\sqrt[k]{k}\right)^2 \xrightarrow[k\to\infty]{} 1^2 = 1$

→ Same thing happens for p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$.

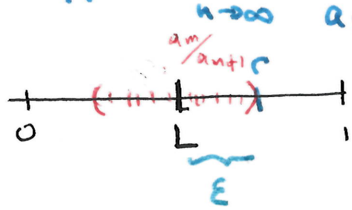
② $\sum_{k=1}^{\infty} \frac{1}{(\ln k)^k}$ converges by Root Test $\sqrt[k]{\frac{1}{(\ln k)^k}} = \frac{1}{\ln k} \xrightarrow[k\to\infty]{} 0$.

③ $\sum_{k=1}^{\infty} \frac{n^2}{2^n}$ ——————— $\sqrt[n]{\frac{n^2}{2^n}} = \frac{\left(\sqrt[n]{n}\right)^2}{2} \to \frac{1^2}{2} = \frac{1}{2} < 1$

Proofs: Use comparison Test with geometric series $\sum_{n=1}^{\infty} r^n$ (or a tail of it) for $r \approx L$ or M.

Ratio Test Proof: Need to show what happens if $L < 1$ or $L > 1$.

(1) Suppose $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L < 1$ Pick $L < r < 1$ & $\varepsilon = r - L$



For $\varepsilon = r - L$ we can find n_0 so that

$L - \varepsilon < \boxed{\frac{a_{n+1}}{a_n} < L + \varepsilon = r}$ for all $n \geq n_0$

So
$$a_{n_0+1} < a_{n_0} r$$
$$a_{n_0+2} < a_{n_0+1} r < a_{n_0} r^2$$
$$a_{n_0+3} < a_{n_0+2} r < a_{n_0} r^3$$
$$\vdots$$
$$a_{n_0+k} < a_{n_0} r^k \text{ for all } k \geq 0$$

$$\boxed{\sum_{m=n_0}^{\infty} a_m} \leq \sum_{m=n_0}^{\infty} a_{n_0} r^m = a_{n_0} \boxed{\sum_{m=n_0}^{\infty} r^m}$$

Since $r < 1$, $\sum_{m=n_0}^{\infty} r^m$ converges (to $\frac{1}{1-r} - \sum_{m=0}^{n_0-1} r^m$)

By comparison Test, $\sum_{m=n_0}^{\infty} a_m$ converges & so does $\sum_{m=1}^{\infty} a_m$ as well.

(2) If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L > 1$, then $a_{k+1} > a_k > 0$ for all k large enough

The terms are increasing & so $a_n \underset{n\to\infty}{\not\to} 0$. The series diverges.

Root Test Proof (1) Suppose $\lim_{n\to\infty} \sqrt[n]{a_n} = M < 1$. We take $M < r < 1$ & $\varepsilon = r - M > 0$. We can find n_0 so that $M-\varepsilon < \boxed{\sqrt[n]{a_n} < M+\varepsilon = r}$ for all $n \geq n_0$

so $a_n < r^n$ for all $n \geq n_0$. Also $a_n > 0$ for $n \geq n_0$.

Then: $\sum_{n=n_0}^{\infty} a_n \leq \sum_{n=n_0}^{\infty} r^n < \infty$ gives convergence of $\sum_{n=n_0}^{\infty} a_n$ by comparison

Same is true for $\sum_{n=1}^{\infty} a_n$.

(2) Pick $M > r > 1$ & $\varepsilon = M - r > 0$ & n_0 so $\sqrt[n]{a_n} > M - \varepsilon = r$ for $n \geq n_0$

Then $a_n > r^n > 0$ for $n \geq n_0$ $\rightsquigarrow$ $\sum_{n=n_0}^{\infty} a_n \geq \sum_{n=n_0}^{\infty} r^n$ diverges since $r > 1$

By comparison test so does $\sum_{n=n_0}^{\infty} a_n$ & hence $\sum_{n=1}^{\infty} a_n$ diverges. $\square$

Application 2: Rearranging a convergent series of non-neg terms does NOT change its sum. (Same is true if only negative terms).

Why? Say we rearrange $\{a_n\}_n$ to $\{b_n\}_n$

Eg: $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + \cdots \rightsquigarrow S_n$ = partial sum

$\sum_{n=1}^{\infty} b_n = a_{10} + a_3 + a_1 + a_{200} + a_5 + a_{99} + \cdots \rightsquigarrow T_n =$ ———

Then $T_4 = a_{10} + a_3 + a_1 + a_{200} \underset{\boxed{a_n \geq 0}}{\leq} a_1 + a_2 + \cdots + a_{200} = S_{200}$

$\sum_{n=1}^{\infty} a_n$ converges so $S_{200} \leq S$. $\rightsquigarrow$ $T_4 \leq S$

Same is true for all T_n: $\left.\begin{array}{l} b_1 = a_{m_1} \\ \vdots \\ b_n = a_{m_n} \end{array}\right\}$ $T_n \leq S_M \leq S$ where $M = \max\{m_1, m_2, \ldots, m_n\}$

- b_n is positive so T_n is increasing & bounded by S, so $\sum_{n=1}^{\infty} b_n$ converges & its sum $\leq S = \sum_{n=1}^{\infty} a_n$

- Symmetrically, (a_n) is a rearrangement of (b_n), The arguments above say $\sum_{n=1}^{\infty} a_n$ converges & its sum S is $\leq \sum_{n=1}^{\infty} b_n$.

We conclude $\sum_{n=1}^{\infty} b_n \le \sum_{n=1}^{\infty} a_n \le \sum_{n=1}^{\infty} b_n$ so $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$.

Rearranging will not change the sum for convergent series with positive terms

⚠ This fails if the series has mixed signs!

Example $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ so $a_n = \frac{(-1)^{n+1}}{n}$

Multiply by $\frac{1}{2}$:

$$\frac{\ln 2}{2} = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n}$$

Add these 2 convergent series term-by-term (columnwise) filling with 0s the odd entries of 2^{nd}. The formula for the sum of the series gives:

$$\begin{array}{cccccccccccc} \ln 2 & = & 1 & -\frac{1}{2} & +\frac{1}{3} & -\frac{1}{4} & +\frac{1}{5} & -\frac{1}{6} & +\frac{1}{7} & -\frac{1}{8} & +\frac{1}{9} & -\frac{1}{10} + \cdots \\ + & & + & & & & & & & & & \\ \frac{\ln 2}{2} & = & 0 & \frac{1}{2} & +0 & -\frac{1}{4} & +0 & +\frac{1}{6} & +0 & -\frac{1}{8} & +0 & +\frac{1}{10} \end{array}$$

$$\boxed{\frac{3}{2}\ln 2} = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \cdots$$

This is a <u>rearrangement</u> of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ with a different sum!

Why?. • All <u>odd</u> terms appear because $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n}$ has no odd denom. $\rightsquigarrow$ 0 in odd spots

• All even terms appear as well: $n = 2m$

If m odd (eg $m=1$): $\frac{(-1)^{2m+1}}{2m}$ disappears from: $\frac{(-1)^{2m+1}}{2m} + \frac{(-1)^{m+1}}{2m} = \frac{-1}{2m} + \frac{1}{2m} = 0$

but appears from $\frac{(-1)^{2(2m)+1}}{2(2m)} + \frac{(-1)^{2m+1}}{2(2m)} = \frac{-1}{4m} - \frac{1}{4m} = \frac{-1}{2m} = \frac{(-1)^{2m+1}}{2m}$

If m even (eg $m=2$): $\frac{(-1)^{2m+1}}{2m}$ appears from $\frac{(-1)^{2(2m)+1}}{2(2m)} + \frac{(-1)^{2(2m)+1}}{2(2m)} = \frac{(-1)}{4m} + \frac{(-1)}{4m}$

$= \frac{-1}{2m} = \frac{(-1)^{2m+1}}{2m}$

So terms for even denominators move from n^{th} place to $2n^{th}$ place.