Lecture L : § 13.8 The alternating series Test. Absolute convergence

Up to now: Studied mostly series of <u>positive terms</u> $\sum_{n=1}^{\infty} a_n$ (with $a_k > 0$ for all $k \geq n_0$)

We have lots of tests for these:

- comparison
- limit comparison
- Integral Test ($f : [0,\infty] \to \mathbb{R}_{\geq 0}$ decr, cont, $f(n) = a_n$)
- Root / Ratio Test.

What about series with both positive & negative terms? These are called <u>alternating</u> series, ie the signs alternate ($+ - + - + -, \ldots$ or $- + - + - +, \ldots$)

Convenient notation $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - a_4 + \cdots$ (with all $a_n \geq 0$)

<u>Example</u>: $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ $(= \ln 2)$ $\left[f(x) = \frac{1}{x}\right]$

$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$ $\left[f(x) = \frac{1}{x^2}\right]$

<u>Note</u>: The alternation in sign gives <u>cancellation</u> which helps for convergence

$\sum_{k=1}^{\infty} \frac{(-1)^k}{2k+1} = 1 - \frac{1}{8} + \frac{1}{5} - \frac{1}{7} + \cdots$ $\left(= \frac{\pi}{4}\right)$ $\left[f(x) = \frac{1}{2x+1}\right]$

Q: Can we decide convergence? Sometimes. (<u>Hard question</u>: find the <u>sum</u>)

<u>Alternating Series Test</u> Pick $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ alternating series with

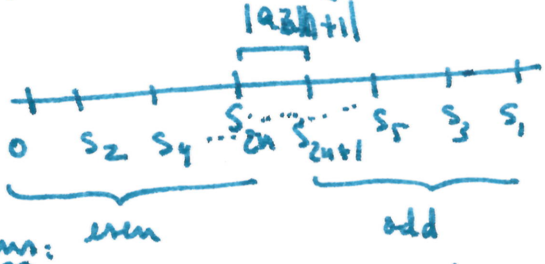
1) $a_k \geq 0$ all positive
2) the $(a_k)_k$ form a decreasing sequence $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_k \geq a_{k+1} \geq \cdots$ (Eg $f : [0,\infty) \to \mathbb{R}_{\geq 0}$ decr. with $f(n) = a_n$ for all n)
3) $\lim_{k \to \infty} a_k = 0$

Then $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ converges

[Same result if (1) & (2) are true for all $n \geq n_0$. (for some fixed n_0)]

<u>Note</u>: All the above examples follow from this test, but we can get the sum

<u>Proof</u>: Recall the argument done for $a_n = \frac{(-1)^{\cdots}}{n}$ (last week)



$S_n = \sum_{k=1}^{n} a_n$ partial sums

<u>Even sums</u>:
$$S_{2n} = \underbrace{(a_1 - a_2)}_{\geq 0} + \underbrace{(a_3 - a_4)}_{\geq 0} + \cdots + \underbrace{(a_{2n-1} - a_{2n})}_{\geq 0}$$
(group them in pairs)

So $S_{2n} \geq 0$ for all n

$S_{2n+1} = S_{2n} + a_{2n+1} \geq S_{2n}$.

Also $S_{2(n+1)} = S_{2n} + \underbrace{(a_{2n+1} - a_{2(n+1)})}_{\geq 0} \geq S_{2n}$

So the partial sums for even number of terms form an <u>increasing</u> sequence

<u>Odd sums</u>:
$$S_{2n+1} = (a_1 - a_2) + \cdots + (a_{2n-1} - a_{2n}) + a_{2n+1} \geq S_{2n} \geq S_2$$
$$= a_1 - \overbrace{(a_2 - a_3)}^{\geq 0} - \overbrace{(a_4 - a_5)}^{\geq 0} - \cdots - \overbrace{(a_{2n} - a_{2n-1})}^{\geq 0} \leq a_1$$

$$S_{2(n+1)+1} = S_{2n+1} - \underbrace{(a_{2(n+1)} - a_{2(n+1)+1})}_{\geq 0} \leq S_{2n+1}$$
so decreasing sequence!

<u>Conclusion</u>:

$S_2 \leq S_4 \leq S_6 \leq \cdots \leq S_1$ bounded above & increasing then <u>convergent</u>

$S_1 \geq S_3 \geq S_5 \geq \cdots \geq S_2$ bounded below & decreasing, so convergent.

But $|S_{2n} - S_{2n+1}| = |a_{2n+1}| \xrightarrow[n \to \infty]{} 0$ so the limit is the same for odd & even sums! □

<u>More examples</u>: (1) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k^2+1}{5-k^2}$ alternating but divergent since $\frac{k^2+1}{5-k^2} \to -1 \neq 0$

(2) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{\ln k}{k}$ $\quad f(x) = \frac{\ln x}{x} \xrightarrow[x \to \infty]{} 0$ (L'Hospital). $\quad f' = \frac{\frac{1}{x} x - \ln x}{x^2} = \frac{1 - \ln x}{x^2} < 0$ for $x > 1$

So it converges by Alternating Series Test.

Q: What if there are signs, but no alternation?

<u>Def</u> $\sum_{k=1}^{\infty} a_k$ is said to be <u>absolutely convergent</u> if $\sum_{k=1}^{\infty} |a_k|$ converges

Prop: An absolutely convergent series must converge.

Proof: Look at $\sum_{k=1}^{\infty} \underbrace{a_k + |a_k|}_{\geq 0} \leq \sum_{k=1}^{\infty} 2|a_k| = 2 \underbrace{\sum_{k=1}^{\infty} |a_k|}_{2L < \infty}$

So $\sum_{k=1}^{\infty}(a_k + |a_k|)$ converges b/c it's positive & bounded (call its sum $= S$)

$\sum_{k=1}^{\infty} |a_k|$ also converges (to L)

By Limit property: $\sum_{k=1}^{\infty}\big((a_k+|a_k|) - |a_k|\big) = \sum_{k=1}^{\infty} a_k$ converges to $S - L$

Advantage: We have a lot of tests for absolute convergence!

Warning: The converse may fail $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$ but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

Series like this one are called conditionally convergent

Curious fact: If $\sum_{n=1}^{\infty} a_n$ is only conditionally convergent, then by rearranging the terms we can make the new series converge to ANY prescribed value or even diverge (see Thm 2 Appendix A13)

More examples: (1) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt[3]{n}}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{1/6}}$ cond. convergent. ($p = \frac{1}{6}$ series)

$f(x) = \frac{1}{\sqrt[6]{x}}$ decr. $f' = \frac{-1}{6x^{7/6}} < 0$. $f(n) \xrightarrow[n\to\infty]{} 0$.

(2) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin^2 n}{n^{9/2}}$ abs convergent $\left|\frac{\sin^2 n}{n^{9/2}}\right| < \frac{1}{n^{9/2}}$ converges b/c $p = \frac{9}{2} > 1$.

(3) $\sum_{n=1}^{\infty} (-1)^{n+1} \ln(\sqrt[n]{n}) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln n}{n}$ converges by Alt Test.

$f(x) = \frac{\ln x}{x}$ $f'(x) = \frac{\frac{1}{x}x - \ln x}{x^2} = \frac{1 - \ln x}{x^2} < 0$

ANSWER $\left|(-1)^{n+1}\frac{\ln n}{n}\right| > \frac{1}{n}$ for $n \geq 3$ so cond. convergent!

Q Is it abs. conv? $a_n = \ln(\sqrt[n]{n})$ $\frac{a_{n+1}}{a_n} = \frac{\frac{\ln(n+1)}{n+1}}{\frac{\ln n}{n}} = \frac{n}{n+1} \frac{\ln(n+1)}{\ln n} \to 1$ Test incond

$\lim_{x\to\infty} \frac{f(x+1)}{f(x)} \sim 0/0 = \lim_{x\to\infty} \frac{f'(x+1)}{f'(x)} = \lim_{x\to\infty} \left(\frac{x}{x+1}\right)^2 \left(\frac{1 - \ln(x+1)}{1 - \ln x}\right) = \lim_{x\to\infty} \left(\frac{1 - \ln(x+1)}{1 - \ln x}\right) \; \infty/\infty = \lim_{x\to\infty} \frac{\frac{-1}{x+1}}{\frac{-1}{x}} \; \infty/\infty = \boxed{1}$. Test inconclusive!

Note $\ln(\sqrt[n]{n})$ is decreasing, so is $\sqrt[n]{n}$ as well so $\frac{a_{n+1}}{a_n} \nearrow 1$