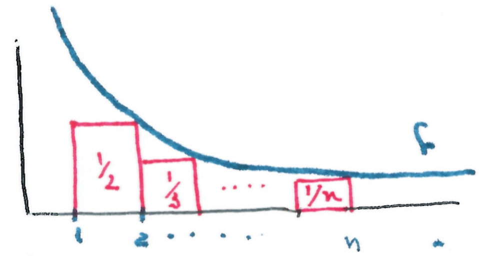
Lecture LI: Appendix A13: Absolute vs Conditional Convergence

Def: A series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ is convergent.

It is conditionally convergent if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

Example: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent.

Claim: The sum is $\ln 2$.



Recall: $f(x) = \frac{1}{x}$ decreasing, positive cont. $f(n) = a_n$

- $b_n = a_1 + a_2 + a_3 + \cdots + a_n - \int_1^n f(x)\,dx$ is bounded & decreasing, so it converges!
- In the case $a_n = \frac{1}{n}$ $\quad b_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \int_1^n \frac{1}{x}\,dx = \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n\right)$

γ = Euler's Constant $= \lim_{n\to\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n\right)$ $\quad (\simeq 0.57721\ldots)$

Equivalently $\lim_{n\to\infty} \underbrace{\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n - \gamma\right)}_{=:\, c_n} = 0$

We say $c_n = o(1)$ if $c_n \xrightarrow[n\to\infty]{} 0$. $\quad$ Equiv: $1 + \frac{1}{2} + \cdots + \frac{1}{n} = \ln n + \gamma + o(1)$

Proof of Claim: Separate Even/Odd Sums.

EVEN

$$S_{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n} \underset{\uparrow}{=} \left(1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}\right) - \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right)$$

regroup even/odd terms

Add & Subtr even terms

$$\overset{\uparrow}{=} \left(1 + \frac{1}{2} + \cdots + \frac{1}{2n}\right) - 2\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right) = \left(1 + \cdots + \frac{1}{2n}\right) - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)$$

$$= \ln 2n + \gamma + o(1) - (\ln n + \gamma + o(1)) = \ln \frac{2n}{n} + o(1) = \ln 2 + o(1)$$

So $S_{2n} \xrightarrow[n\to\infty]{} \ln 2$

ODD: $S_{2n+1} = S_{2n} + \frac{1}{2n+1} \xrightarrow[n\to\infty]{} \ln 2 + 0 = \ln 2$.

Both even & odd partial sums have the same limit $= \ln 2$. $\square$

Claim 2: The rearrangement $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots = \frac{3}{2}\ln 2$

has a different sum! (see Lecture XLVII)

<u>Last week</u>: This phenomenon does <u>not</u> happen for convergent positive series!

Q: Does it happen for absolutely convergent series? <u>A</u> NO.

<u>THM 1</u>: An absolutely convergent series with sum S will have the <u>same</u> sum for any rearrangement.

<u>THM 2</u>: Assume $\sum_{n=1}^{\infty} a_n$ is conditionally convergent. Then, its terms can be rearranged to give a convergent series with <u>any</u> <u>prescribed sum</u> S, in $\mathbb{R}$ or a series that is divergent with sum $+\infty$ or $-\infty$

<u>Main technique</u>. Given $\sum_{n=1}^{\infty} a_n$, define 2 positive sequences:

$$\boxed{p_n = \frac{|a_n| + a_n}{2}} = \begin{cases} a_n & \text{if } a_n \geq 0 \\ 0 & \text{if } a_n < 0 \end{cases} \quad \& \quad \boxed{q_n = \frac{|a_n| - a_n}{2}} = \begin{cases} 0 & \text{if } a_n \geq 0 \\ -a_n & \text{if } a_n \leq 0 \end{cases}$$

<u>Lemma</u>: (1) If $\sum_{n=1}^{\infty} a_n$ converges conditionally, then $\sum_{n=1}^{\infty} p_n$ & $\sum_{n=1}^{\infty} q_n$ <u>both</u> diverge

(2) If $\sum a_n$ converges absolutely then both $\sum_{n=1}^{\infty} p_n$ & $\sum_{n=1}^{\infty} q_n$ converge & furthermore $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} p_n - \sum_{n=1}^{\infty} q_n$.

<u>Proof</u>: By construction $a_n = p_n - q_n$ & $|a_n| = p_n + q_n$

We know: convergent series can be add/substracted term by term.

(1) If $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges then $\sum_{n=1}^{\infty} p_n$ must diverge

(otherwise $\sum |a_n| = \sum (2p_n - a_n) = 2\sum_{n=1}^{\infty} p_n - \sum_{n=1}^{\infty} a_n$ will converge, but it doesn't)

For $\sum_{n=1}^{\infty} q_n$ it's the same idea.

(2) If $\sum_{n=1}^{\infty} a_n$ & $\sum_{n=1}^{\infty} |a_n|$ both converge, then $\sum_{n=1}^{\infty} p_n = \frac{1}{2}\left(\sum_{n=1}^{\infty} |a_n| + \sum_{n=1}^{\infty} a_n\right)$ converges. Same idea proves $\sum_{n=1}^{\infty} q_n$ converges.

Then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (p_n - q_n) = \sum_{n=1}^{\infty} p_n - \sum_{n=1}^{\infty} q_n$ (difference of conv. series)

<u>Proof of THM 1</u>: $\sum |a_n|$ is positive & convergent (to S). We know any rearrangement of it has the same sum.

Call $\sum_{n=1}^{\infty} b_n$ this rearrangement of $\sum a_n$. Then $\sum_{n=1}^{\infty} |b_n|$ is abs. convergent with sum $= S$.

By Lemma 1, write $\begin{cases} p_n = \dfrac{|a_n|+a_n}{2} \\ q_n = \dfrac{|a_n|-a_n}{2} \end{cases}$ & $\begin{cases} \tilde{p}_n = \dfrac{|b_n|+b_n}{2} & \text{rearr of } p_n \text{ (all} \geq 0) \\ \tilde{q}_n = \dfrac{|b_n|-b_n}{2} & \text{rearr of } q_n \end{cases}$

we know both $\tilde{p}_n$ & $\tilde{q}_n$ converge & $\boxed{\sum_{n=1}^{\infty} b_n} = \sum_{n=1}^{\infty} \tilde{p}_n - \sum \tilde{q}_n \overset{(*)}{=} \sum_{n=1}^{\infty} p_n - \sum_{n=1}^{\infty} q_n \overset{\text{LEMMA}}{=} \boxed{\sum_{n=1}^{\infty} a_n}$.

(*) Rearr of pos. convergent series have equal sum!

We get $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n$ so the sum is preserved!

Proof of THM 2 (Riemann) ① Fix the desired sum s in $\mathbb{R}$ we want

We know by Lemma that $\sum_{n=1}^{\infty} p_n$ & $\sum_{n=1}^{\infty} q_n$ both diverge. $(=\infty)$

Start by writing down p's until the first one of the partial sum $p_1+p_2+\cdots+p_{n_1} \geq s$.

($p_i = a_i$ for $a_i > 0$). Then, continue with q's $(=-a_i)$ until for the first time,

$$p_1+p_2+\cdots+p_{n_1}-q_1-q_2-\cdots-q_{m_1} \leq s.$$

(if $p_n>0$ $q_n=0$; if $q_n>0$ $p_n=0$; so things are counted ONLY once!)

Then continue with p's until we pass s for the first time: $p_1+\cdots+p_{n_1}-q_1-\cdots-q_{m_1}+p_{n_1+1}+\cdots+p_{n_2} \geq s$

[Each step can be achieved because $\sum_{n=1}^{\infty} p_n = \sum_{n=1}^{\infty} q_n = \infty$]

Claim 1: This gives a rearrangement of a_n.

Claim 2: The rearrangement has sum s because $a_n \to 0$ so $p_n \to 0$ & $q_n \to 0$ & by construction each $n_1, n_2, \ldots, m_1, m_2, \ldots$ are chosen so that all partial sums get closer to s as we move along)

② To make $\sum b_n = +\infty$, write down $p_1+\cdots+p_{n_1} \geq 1$

then insert $-q_1$, then continue w/ ps $p_1+\cdots+p_{n_1}-q_1+p_{n_1+1}+\cdots+p_{n_2} \geq 2$

Then " $-q_2$, and so on.

③ To make $\sum b_n = -\infty$, reverse the role of p & q above. That is

$-(q_1+\cdots+q_{n_1}) \leq -1$, $\quad -q_1-\cdots-q_{n_1}+p_1-q_{n_1+1}-\cdots-q_{n_2} \leq -2$, etc

Fun exercises: Show, via $1+\frac{1}{2}+\cdots+\frac{1}{n} = \ln n + \gamma + o(1)$

(1) $\underbrace{1+\frac{1}{3}+\frac{1}{5}}_{3+} \underbrace{-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}}_{3-} + \cdots = \ln 2$

(2) $\underbrace{1+\frac{1}{3}+\frac{1}{5}}_{3+} \underbrace{-\frac{1}{2}-\frac{1}{4}}_{2-} + \frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}-\frac{1}{8}+\cdots = \frac{1}{2}\ln 6$

(3) $\underbrace{1}_{1+} \underbrace{-\frac{1}{2}-\frac{1}{4}}_{2-} + \underbrace{\frac{1}{3}}_{1+} - \frac{1}{6} - \frac{1}{8} = \frac{1}{2}\ln 2$

(4) $\underbrace{1}_{1+} \underbrace{-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\frac{1}{8}}_{4-} + \frac{1}{3} - \frac{1}{10} - \frac{1}{12} - \frac{1}{14} - \frac{1}{16} + \cdots = 0.$

Fun fact: The rearrangement of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ with $p+$, $q-$ terms pattern converges to $\ln\left(2\sqrt{\frac{p}{q}}\right)$ (checks out for $p=q=1$, $p=3, q=2$ & all examples above).