Lecture LIII : §14.1 Introduction to power series
§14.2 The interval of convergence of a power series

GOAL: Understand expressions of functions in power series in x, ie

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots\cdots + a_n x^n + \cdots = \sum_{n=0}^{\infty} a_n x^n .$$

$\boxed{\begin{array}{l} a_n = \text{constants} \\ x = \text{variable} \end{array}}$

(1) Given a known function (e.g a soln to a differential eqn.), can we write it as a power series?

(2) If so, is the equality valid for any x?

Examples: ① $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots\cdots = \dfrac{1}{1-x}$ for $|x|<1$.

(note: cannot evaluate (RHS) at $x=1$)

② $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots\cdots$ Region of validity = ??

(came from $\begin{array}{l} y' = y \\ y(0) = 1 \end{array}$ & trying to write a soln as $\sum_{n=0}^{\infty} a_n x^n$)
differentiate term by term to get $a_n = \frac{1}{n!}$

③ $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \cdots\cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{x^n}{n}$ Region of validity = ??

[works for $x=1$ (Lecture LI); $x=0$]

(Know $\dfrac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$ & integrate power series)

④ $\tan^{-1}(x) = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \cdots\cdots = \sum_{n=0}^{\infty} \dfrac{x^{2n+1}}{2n+1} (-1)^n$ Region of validity = ??

⑤ $\left.\begin{array}{l} \sin(x) = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \cdots \\ \cos(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \cdots \end{array}\right\}$ Trying to solve $y'' + y = 0$ as power series, we get $a\ \text{series}_1 + b\ \text{series}_2 = a \sin x + b \cos x$.

Region of validity = ??

Note: Given a power series $\sum a_n x^n$, whenever we have a specific value of the variable x for which a series converges, we can use it to define a function

(Domain = Region of validity in examples above).

The function may not be defined everywhere!

Example ① The geometric series $\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$ converges ABSOLUTELY for $|x|<1$. It diverges at $x = \pm 1$ & for $|x|>1$.

Conclusion: The geom series defines a function on $(-1,1)$.

Note: In its domain, we know the function has the simpler formula $\frac{1}{1-x}$

Example ② $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges ABSOLUTELY for all x [• For $x=0$ sum $= 1+0+0+\cdots = 1$

• For $x \neq 0$, show $\sum_{n=0}^{\infty} \underbrace{\frac{|x|^n}{n!}}_{=a_n}$ converges by Ratio Test: $\frac{a_{n+1}}{a_n} = \frac{|x|}{n+1} \xrightarrow[n\to\infty]{} 0 < 1$]

So Domain $= \mathbb{R} = (-\infty, \infty)$.

Example ③ $\sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}$ ~~converges absolutely~~ on $(-1,1)$

$\begin{cases} x=0 & \text{converges to } 0 \text{ \& abs conv} \\ x=1 & \text{converges cond to } \ln 2 \\ x=-1 & \text{diverges} \end{cases}$

Why? • For $0<x<1$ we get an alt. series. • $a_n = \frac{x^n}{n} > 0$ for all n.

• $a_n \xrightarrow[n\to\infty]{} 0$ ($x<1$ so $x^k \xrightarrow[k\to\infty]{} e^{-\infty} = 0$)

• (a_n) is ultimately decreasing!

By AST, the series converges. $\frac{a_{n+1}}{a_n} = \underbrace{x \frac{n}{n+1}}_{\substack{\downarrow n\to\infty \\ x<1}} < 1$ for n big enough

• For $-1<x<0$: $\sum_{k=1}^{\infty} \frac{(-1)(-x)^k}{k} = -\sum_{k=1}^{\infty} \frac{\overbrace{(-x)}^{>0}{}^k}{k}$

& $0 < \sum_{k=1}^{\infty} \frac{(-x)^k}{k} \leq \sum_{k=1}^{\infty} \underbrace{(-x)}_{<1}{}^k = \frac{1}{1+x} < \infty$ so it converges by comparison test!

• Diverges at $x=-1$ because we get $-\sum_{k=1}^{\infty} \frac{1}{k}$ (harmonic series)

• Converges at $x=1$ " " " $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \ln 2$

So Domain includes $(-1, 1]$

Q: What happens on $[1,\infty)$ & $(-\infty,-1)$? So Domain $= (-1,1]$

diverges (diverges because series $= -\sum_{k=1}^{\infty} \frac{|x|^k}{k}$ & $\frac{|x|^k}{k} \to \infty$)

because

$(-1)^{n+1} \frac{x^n}{n} = A_n \xrightarrow[n\to\infty]{} \pm\infty$ ($\frac{x^n}{n}$ behaves like $\frac{e^n}{n}$)

Q: To what extent are these ranges of convergence typical?

Useful Lemma (1) If a power series $\sum_n a_n x^n$ converges at x_1 & $x_1 \neq 0$, then it converges absolutely for all x with $|x| < |x_1|$.

(2) If the power series diverges at x_1, then it diverges for all x with $|x| > |x_1|$

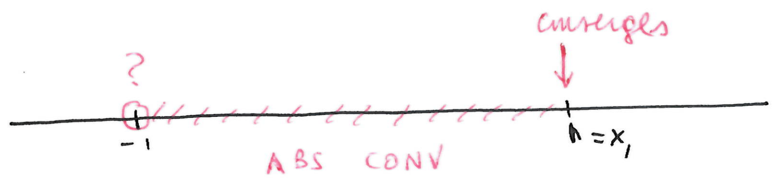
Example: $\displaystyle\sum_{k=1} \frac{(-1)^{k+1} x^k}{k}$

(1) Pick. $x_1 = 1$ $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges so Lemma says we converge absolutely for $|x| < 1$

(2) Pick $x_2 = -1$ $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(-1)^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)}{k}$ diverges, so Lemma says we diverge for $|x| > 1$

Missing cases: $|x| = 1$ so $x = 1$ & $x = 1$

For (1)



as our calculations predicted

For (2)

Next Time: Proof of Lemma.