Lecture LIV: §14.2 The interval of convergence (cont.)

Recall: $f(x) = a_0 + a_1 x + a_2 x^2 + \dots = \sum_{i=0}^{\infty} a_i x^i$ is a power series in x

It is a function of x.

Q1: Domain of f? (It includes $x=0$!) $\boxed{\text{Domain} = \text{Interval of Convergence}}$

Q2: Can we find an elementary function representing this series? ⇝ Integrate / Differentiate / Diff'l eqns

Q3: Can we find power series representation for any function? ⇝ Taylor Series

Ex: $f(x) = \sum_{n=0}^{\infty} x^n$ is defined for $-1 < x < 1$ & it equals $\frac{1}{1-x}$ in this range.

§1 Interval of Convergence:

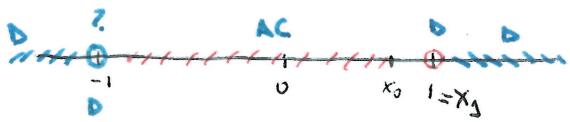
Very Useful Theorem: Consider $f(x) = \sum_{i=0}^{\infty} a_i x^i$:

① If $f(x)$ converges at x_0 & $x_0 \neq 0$ then it converges ABSOLUTELY for all x with $|x| < |x_0|$

② If $f(x)$ diverges at x_1, then it diverges for all x with $|x| > |x_1|$.

Ex above:
- $f(x)$ diverges at $x_1 = 1$ so diverges for $|x| > |1| = 1$
- " converges for $|x_0| < 1$ " converges abs for $|x| < |x_0|$

only missing $x = -1$. diverges!



Ex 2: $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges absolutely for any x. $(= e^x)$

Why? Fix $x_0 > 0$ & use Ratio Test $\frac{b_{n+1}}{b_n} = \frac{\frac{1}{(n+1)!} x_0^{n+1}}{\frac{1}{(n)!} x_0^n} = \frac{x_0}{n+1} \xrightarrow[n\to\infty]{} 0 < 1$ for all x_0

So we have convergence for all x_0.

The Thm above then says we converge absolutely for any x in $\mathbb{R}$.

(for example, take $x_0 = |x| + 1$).

Usual situations: Abs convergence everywhere, only at 0, or we can find x_0 with $|x_0|$ minimal where we diverge (this will be the radius of convergence) Domain = Interval of Conv.

Proof of Theorem:

① Suppose $f(x)$ converges for $x_0 \neq 0$, then $a_n x_0^n \xrightarrow[n\to\infty]{} 0$. This means that eventually (for $n \geq n_0$) we must have $|a_n x_0^n| < 1$ ($\varepsilon = 1$ in def of $\lim_{n\to\infty}$)

But then, if $|x|<|x_0|$, $|a_n x^n| = \left|a_n \left(\frac{x}{x_0} x_0\right)^n\right| = \underbrace{|a_n x_0^n|}_{<1} \left|\frac{x}{x_0}\right|^n < \left|\frac{x}{x_0}\right|^n$ for $n \geq n_0$

In particular, set $r = \left|\frac{x}{x_0}\right| < 1$. gives

$$\sum_{n=n_0}^{\infty} |a_n x^n| \leq \sum_{n=n_0}^{\infty} r^n = r^{n_0} \sum_{n=0}^{\infty} r^n = \frac{r^{n_0}}{1-r} \rightsquigarrow \sum |a_n x^n| \text{ converges by Comparison!}$$

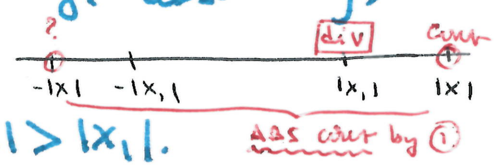
So $\sum_{n=0}^{\infty} |a_n x^n| = |a_0| + |a_1 x| + \cdots + |a_{n_0-1} x^{n_0-1}| + \sum_{n=n_0}^{\infty} |a_n x^n|$ also converges!

Conclusion: $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x|<|x_0|$.

② The argument is simpler, goes through Comparison with ①. Start with arguing by contradiction.

Assume for some x with $|x| > |x_1|$ we have convergence.

Then by ①, the series $\sum_{n=0}^{\infty} a_n x_1^n$ would converge absolutely, so it would be convergent. This is a contradiction!



Conclusion: $\sum_{n=0}^{\infty} a_n x^n$ diverges for all x with $|x| > |x_1|$.

Consequence: Given $\sum_{n=0}^{\infty} a_n x^n$ precisely only one of the following occurs:

(1) The series converges only for $x=0$ $[R=0]$

(2) ——— for all x (& absolutely) $[R=+\infty]$

(3) There is a positive real number R (radius of convergence) for which the series:
- converges absolutely for $|x|<R$
- diverges for $|x|>R$

$[0<R<\infty]$

⚠ Nothing can be said for $x = \pm R$. In each example we must treat these 2 points separately.

Intervals of Convergence: (1) $[0,0]$, (2) $\mathbb{R} = (-\infty,\infty)$, (3) 4 options $[-R,R]$, $(-R,R)$, $[-R,R)$, $(-R,R]$

§2 Examples $\boxed{\text{ROC computed with ratio test / root test}}$

(1) $\sum_{n=0}^{\infty} n!\, x^n$ $\boxed{\text{ROC} = 0}$

Ratio Test, $x \neq 0$: $\left|\frac{(n+1)!\, x^{n+1}}{n!\, x^n}\right| = |x|(n+1) \xrightarrow[n\to\infty]{} \infty$ for all $x \neq 0$. So no abs conv. for $x \neq 0$.

(2) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ ROC $= +\infty$ by Ratio Test

(3) (a) $\underline{(-R,R)}$: $\sum_{n=0}^{\infty} x^n$ $(R=1)$

(b) $\underline{(-R,R]}$: $\sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^n}{n}$ $(R=1)$

(c) $\underline{[-R,R)}$: $\sum_{n=0}^{\infty} \frac{x^n}{n+1}$ $(R=1)$

ROC by Ratio Test: $\left|\frac{b_{n+1}}{b_n}\right| = \frac{|x|^{n+1}}{n+1} \Big/ \frac{|x|^n}{n} = \frac{n}{n+1}|x| \xrightarrow[n\to\infty]{} |x|$

- If $|x|<1$ we get abs. convergence
- If $|x|>1$ we get divergence of abs. series

$\Big\}$ we conclude $\underline{R=1}$

- $\underline{x=1}$: $\sum_{n=0}^{\infty} \frac{1}{n+1}$ harmonic series, so it diverges
- $\underline{x=-1}$: $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \ln 2$ (converges by AST)

(d) $\underline{[-R,R]}$: $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

ROC by Ratio Test: $\left|\frac{b_{n+1}}{b_n}\right| = \frac{|x|^{n+1}}{(n+1)^2} \Big/ \frac{|x|^n}{n^2} = \left(\frac{n}{n+1}\right)^2 |x| \xrightarrow[n\to\infty]{} |x|$

- If $|x|<1$ we get abs. convergence
- If $|x|>1$ we get divergence of abs series

$\Big\}$ we conclude $\underline{R=1}$

- $\underline{x=1}$ $\sum_{n=0}^{\infty} \frac{1}{n^2}$ converges (p-series with $p=2$)
- $x=-1$ $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2}$ " " absolutely, so it converges

$\underline{Q}$: What if we have $f_{(x)} = \sum_{n=0}^{\infty} a_n (\underbrace{x-c}_{=z})^n$?

1. Compute radius of conv. for $\sum_{n=0}^{\infty} a_n z^n$: R $\rightsquigarrow$ ROC of $f_{(x)}$ is also R.
2. Interval of conv for $\sum_{n=0}^{\infty} a_n z^n$: $(-R,R]$ $\rightsquigarrow$ IOC for $f(x)$ is $(-R+c, R+c]$ (shift by c)