Lecture LV: §14.3 Differentiation & Integration of power series
§14.4 Taylor series & Taylor's Formula

Modelled on polynomial behavior, we ask:

Q: Can we differentiate / integrate power series term-by-term?

A: YES, but the analysis is very delicate (need "uniform convergence" A15)

Theorem: Pick $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with radius of convergence $R>0$ (so $f(x)$ is a function on $(-R,R)$). Then:

① $f(x)$ is continuous on $(-R,R)$

② $f(x)$ is differentiable on $(-R,R)$ & we get $f'(x)$ by term-by-term differentiation:
$$f'(x) = \Big(\sum_{n=0}^{\infty} a_n x^n\Big)' = \sum_{n=0}^{\infty} (a_n x^n)' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$= a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

③ $f(x)$ can be integrated term-by-term on $(-R,R)$:
$$\int_0^x f(t)\,dt = \int_0^x \sum_{n=0}^{\infty} a_n t^n\,dt = \sum_{n=0}^{\infty} \frac{a_n t^{n+1}}{n+1}\Big|_0^x = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}.$$

Note: Can repeat the process in ② and get that $f(x)$ is infinitely differentiable on $(-R,R)$ & $\frac{d^k f}{dx^k} = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)\,a_n x^{n-k}$. for $k=1,2,\ldots$

Application 1: Use this to get new expressions for series.

Example ① $f(x) = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n$ for x in $(-1,1)$ (abs conv)

We integrate & get
$$\ln(1+x) = \int_0^x \sum_{n=0}^{\infty} (-t)^n\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{n+1}\Big|_0^x$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

This equality is valid on $(-1,1)$!

Q: Can we push this to $x=\pm 1$?

A: For $x=1$: $\ln(2) \stackrel{?}{=} 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ True!

For $x=-1$: $\underset{-\infty}{\overset{\shortparallel}{\ln(0)}} \stackrel{?}{=} -1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{-1}{n}$ diverges in both sides

Example ②: $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ has $R = +\infty$ (by Ratio Test).

Compute f' term-by-term:

$$f' = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \right)' = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} = f$$

So $f_{(x)}$ is a solution to the diff'l equation $y' = y$ & initial condition $f(0) = 1 + 0 = 1$

We know another solution $g(x) = e^x$. with $g(0) = 1$.

By uniqueness: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ & this is valid for all x in $\mathbb{R}$.

Example ③: $\sum_{n=1}^{\infty} n^2 x^n = x + 4x^2 + 9x^3 + 16x^4 + \cdots$

Q: What elementary function does this represent (if any)?

Claim: ROC $= 1$.

Why? Use Ratio Test: $\frac{(n+1)^2 |x|^{n+1}}{n^2 |x|^n} = |x| \left(\frac{n+1}{n}\right)^2 \xrightarrow[n \to \infty]{} |x|$

So converges if $|x| < 1$ & diverges if $|x| > 1$, so ROC $= 1$.

So $f(x) = \sum_{n=1}^{\infty} n^2 x^n = x \underbrace{\left(1 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \cdots\right)}_{= g(x)}$ is defined on $(-1, 1)$

(divergent for $x = \pm 1$ since $|n^2| \underset{n \to \infty}{\not\to} 0$)

- $g(x) = \sum_{n=0}^{\infty} (n+1)^2 x^n$ has also ROC $= 1$
- We integrate $g(x)$ term-by-term & get a new function $h(x) = \int_0^x g(t)\,dt$ with ROC $= 1$

$$h(x) = \sum_{n=0}^{\infty} \int_0^x (n+1)^2 t^n \, dt = \sum_{n=0}^{\infty} \frac{(n+1)^2 x^{n+1}}{n+1} = \sum_{n=0}^{\infty} (n+1) x^{n+1}.$$

$$= x + 2x^2 + 3x^3 + \cdots = x \underbrace{\left(1 + 2x + 3x^2 + \cdots\right)}_{j(x) = \sum_{n=0}^{\infty} (n+1) x^n \text{ has ROC} = 1}.$$

- We integrate $j(x)$ term-by-term & get a new function $P(x)$ with ROC $= 1$

$$P(x) = \int_0^x j(t)\,dt = \int_0^x \sum_{n=0}^{\infty} (n+1) t^n \, dt = \sum_{n=0}^{\infty} \int_0^x (n+1) t^n \, dt = \sum_{n=0}^{\infty} x^{n+1} = x \sum_{n=0}^{\infty} x^n$$

So $P(x) = \frac{x}{1-x}$ (because ROC $= 1$)

- Now, we <u>reverse</u> the process by <u>Fundamental Thm of Calculus</u>!

$$j(x) = P_{(x)}' = \left(\frac{x}{1-x}\right)' = \frac{1(1-x) - x(-1)}{(1-x)^2} = \frac{1}{(1-x)^2} \qquad \left(j_{(0)} = 0 \neq \frac{1}{(1-0)^2}\right)$$

$$h(x) = x\, j(x) = \frac{x}{(1-x)^2}$$

$$g(x) = h'(x) = \frac{1(1-x)^2 - x\,2(1-x)(-1)}{(1-x)^4} = \frac{1-x+2x}{(1-x)^3} = \frac{1+x}{(1-x)^3}$$

So $f(x) = x\, g(x) = \dfrac{x(1+x)}{(1-x)^3}$ can be expressed as a power series on $(-1,1)$, namely $\sum_{n=1}^{\infty} n^2 x^n$.

<u>Application 2</u>: Taylor series of $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with ROC $= R > 0$.

We know $f', f'', \dots, f^{(k)}, \dots$ all exist & are power series with ROC $= R$

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \cdots$$

$$f''(x) = 2a_2 + 3!\, a_3 x + 4.3\, a_4 x^2 + 5.4\, a_5 x^3 + \cdots$$

$$f'''(x) = 3!\, a_3 + 4.3.2\, a_4 x + 5.4.3\, a_5 x^2 + \cdots$$

<u>In general</u>: $f^{(n)}(x) = n!\, a_n +$ Terms containing x as a factor ($=$ series with ROC R)

So $f^{(n)}(0) = n!\, a_n + 0 = n!\, a_n$ gives

$$\boxed{\text{Taylor coeff}: \ a_n = \frac{f^{(n)}(0)}{n!} \text{ for all } n = 0, 1, \dots}$$

($0! = 1$ convention)

<u>Conclusion</u>: If a function f can be represented by a power series with ROC $= R > 0$ then, the series <u>must</u> be the <u>Taylor series</u> (around $x=0$); $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$, i.e.

$$f(x) = f(0) + f'(0)\, x + \frac{f''(0)}{2} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + \cdots$$

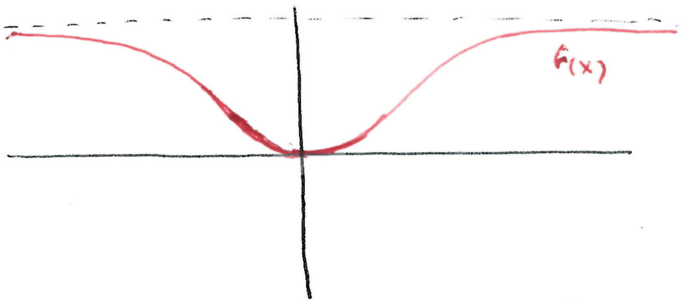
Q: Can we reverse this?

- Given a function f differentiable up to <u>any</u> order at $x=0$, we can write down the Taylor series $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$. Does this series represent $f(x)$?

<u>A</u>: NO!

<u>Ex</u>: $f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & x = 0 \end{cases}$ Can compute all $f^{(n)}(0)$ by definition & get $f^{(n)}(0) = 0$ for all n.

So Taylor series $= \sum_{n=0}^{\infty} 0\, x^n$. But f is <u>NOT</u> $= 0$, it's just <u>very flat</u>.



$$f'(0) = \lim_{x \to 0} \frac{e^{-1/x^2}}{x} = 0 \quad \text{by L'Hosp.}$$

$$f'(x) = \frac{e^{-1/x^2}}{x^3} \quad \text{for } x \neq 0$$

$$\text{So } f''(x) = \lim_{x \to 0} \frac{e^{-1/x^2}}{x^4} = 0 \quad \text{by L'Hosp, etc.}$$

[Ex 42 §12.3]

<u>Easy extension</u>: Taylor series of f around a fixed pt c in the domain of f.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n .$$

<u>Natural question</u>: When can we represent f by its Taylor series around a point c in the domain of f?

In other words: $f(x) \overset{?}{=} \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n$

$\boxed{\text{Focus on the case } c=0.}$ Assume $R = \text{ROC}$ of Taylor series is >0.

$\rightsquigarrow$ Lagrange remainder Formula.

IDEA: Work with <u>partial sums</u> (also known as Truncations of the Taylor series).
& <u>estimate the error</u> in the approximation.

$$S_N(x) = f(0) + f'(0)\boxed{x} + \frac{f''(0)}{2}\boxed{x^2} + \frac{f'''(x)}{3!}\boxed{x^3} + \dots + \frac{f^{(N)}(0)}{N!}\boxed{x^N}$$

polynomial of degree $\leq N$ in x.

Call $\boxed{R_N(f)(x) := S_N(x) - f(x) \quad \text{Remainder.}}$

<u>Prop</u> $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ converges to $f(x)$ if and only if $|R_N(f)(x)| \to 0$ for every x in $(-R, R)$.

⚠ This is only useful if we can give a formula for $R_N(f)(x)$.

- Remainder Formula at 0: $R_N^{(f)}(x) = \frac{f^{(N+1)}(b)}{(N+1)!} x^{N+1}$ for b between 0 & x
- Remainder Formula at c: $R_N^{(f)}(x) = \frac{f^{(N+1)}(b)}{(N+1)!} (x-c)^{N+1}$ for b between c & x.