Lecture LVI: §14.4 (cont.) Taylor series & Taylor's formula.

Recall: Taylor series of a function f centered at c. (c in the domain of f)

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n \quad \rightsquigarrow \text{Need } f \text{ to be differentiable up to any order at } x=c.$$

Q: When can we say $f(x) =$ Taylor series at c (for c in the domain of f)?

A: Last time: not always works! $\boxed{\text{Set } c=0}$

IDEA: Truncate the Taylor series = work with the partial sums of the Taylor series & estimate the error in the approximation.

Write: $S_N(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$

degree $\leq n$ Taylor polynomial

Write $R_N(f)(x) = f(x) - S_N(x)$ Remainder.

Prop: $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$ with ROC $=R>0$ converges to $f(x)$ if and only if

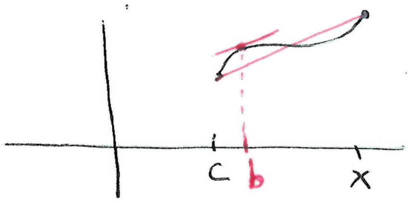
$$|R_N(f)(x)| \xrightarrow[N\to\infty]{} 0 \quad \text{for every } x \text{ in } (-R,R).$$

⚠ The N_0 that works for one x may not be enough for another x.

- This is only useful if we can give a formula for $R_N(f)(x)$.
- Lagrange Remainder Formula: $R_N(f)(x) = \frac{f^{(N+1)}(b)}{(N+1)!}x^{N+1}$ for b between 0 & x
- Other centers? $\boxed{R_N(f)(x) = \frac{f^{(N+1)}(b)}{(N+1)!}(x-c)^{N+1}}$ for b between c & x

Q: Why should we expect this?

$\underline{N=0}$ $\quad R_0(f)(x) = f(x) - f(c) \overset{?}{=} f'(b)(x-c)$ This is Mean Value Thm!



$$\frac{f(x)-f(c)}{x-c} = \text{slope of secant line}$$
$$= \text{slope of Tang line for some } b \text{ between } c \text{ \& } x.$$

. The proof for higher N will involve Mean Value theorem as well.

Reason 2: Say $f_{(x)} = \sum_{n=0}^{\infty} \frac{f^{(n)}_{(c)}}{n!}(x-c)^n$ with ROC >0.

Then
$$f(x) - S_N(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}_{(c)}}{n!}(x-c)^n - \sum_{n=0}^{N} \frac{f^{(n)}_{(c)}}{n!}(x-c)^n$$
$$= \sum_{n=N+1}^{\infty} \frac{f^{(n)}_{(c)}}{n!}(x-c)^n = \frac{f^{(N+1)}_{(c)}}{(N+1)!}(x-c)^{N+1} + \dots\dots$$

Our formula replaces this tail ↑ by $\frac{f^{(n+1)}_{(b)}}{(N+1)!}(x-c)^{N+1}$ (which looks very similar to the 1st Term in the tail.).

§2 Applications

① Show $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

. Know by Ratio Test that ROC $=\infty$.

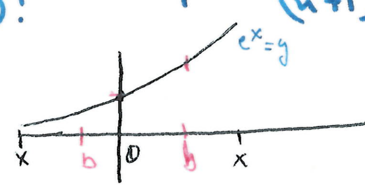
Q: What's the Taylor series of e^x at $x=0$?

$$\left.\begin{array}{lll} f_{(x)} = e^x & \rightsquigarrow & f_{(0)} = e^0 = 1 \\ f'_{(x)} = e^x & \rightsquigarrow & f'_{(0)} = 1 \\ \vdots & & \\ f^{(n)}_{(x)} = e^x \text{ for all } n & \rightsquigarrow & f^{(n)}_{(0)} = 1. \end{array}\right\} \text{(RHS) } \underline{\text{is}} \text{ Taylor series}$$

Remainder? $\rightsquigarrow$ $|R_N(x)| = \left|\frac{e^b}{(n+1)!} x^{n+1}\right| = \frac{|x|^{n+1}}{(n+1)!} |e^b|$

If $x>0$ $|e^b| < |e^x|$

If $x<0$ $|e^b| < |e^0| = 1$



$\rightsquigarrow$ Bound $|f^{(n+1)}_{(b)}| \le M$. where M is independent of b, only depends on center & x

In both cases: $|R_N(x)| \le \frac{|x|^{n+1}}{(n+1)!} \underbrace{\max\{e^x, 1\}}_{=M}$ & $\frac{|x|^{n+1}}{(n+1)!} \xrightarrow[n\to\infty]{} 0$ (b/c $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is abs conv)

By Squeeze Thm: $|R_n(x)| \xrightarrow[n\to\infty]{} 0$ for each fixed x.

Conclusion: e^x can be expressed by its Taylor series (as we already knew!)

(2) Taylor series for $\sin(x)$:

$f(x) = \sin(x) \rightsquigarrow f(0)=0$
$f'(x) = \cos(x) \rightsquigarrow f'(0)=1$
$f''(x) = -\sin(x) \rightsquigarrow f''(0)=0$
$f'''(x) = -\cos(x) \rightsquigarrow f'''(0)=-1$
$f^{(4)}(x) = \sin(x) \rightsquigarrow f^{(4)}(0)=0$

& Repeat from then on. So the Taylor series for $f(x)$ is $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}}$$

$\rightsquigarrow$ only ODD powers ($\sin(x)$ is an odd function)

. ROC = ?

$$\left|\frac{x^{2n+3}}{(2n+3)!}\right| \Big/ \left|\frac{x^{2n+1}}{(2n+1)!}\right| = \frac{|x|^2}{(2n+3)(2n+2)} \xrightarrow[n\to\infty]{} 0 < 1 \text{ for any } x\neq 0 \text{ fixed.}$$

So $\boxed{ROC = +\infty}$

. Remainder Formula:

$$|R_n(x)| \leq \frac{|f^{(n+1)}(b)|}{(n+1)!}|x|^{n+1} \leq \frac{1}{(n+1)!}|x|^{n+1} \xrightarrow[n\to\infty]{} 0$$

($f^{(n+1)}(b) = \pm\sin(b), \pm\cos(b)$) for any fixed x

(Why? $(n+1)^{st}$ term of series for $e^{|x|}$, so it $\xrightarrow[n\to\infty]{} 0$).

Conclusion: $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ on $\mathbb{R}$

(3) Taylor series for $\cos(x)$:

$$\cos(x) = (\sin(x))' = \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}\right)' \underset{\text{term-by-term}}{=} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \text{ with } ROC = +\infty$$

$\rightarrow$ ONLY EVEN POWERS

Also: (RHS) is the Taylor series of $\cos(x)$ by uniqueness.

(4) Taylor series for e^{-x^2}:

$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ has ROC $= +\infty$ $\rightsquigarrow$ $\boxed{\text{Substitute } z = -x^2}$

$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$ has ROC $= +\infty$ so it must be the Taylor series

. Can use this to approximate

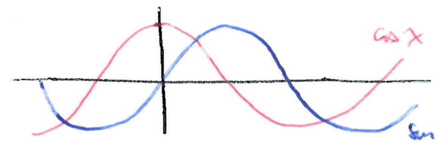
$$\int_0^1 e^{-x^2}dx \underset{\text{integrate term-by term}}{=} \sum_{n=0}^{\infty} \int_0^1 \frac{(-1)^n x^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\, n!}\Bigg|_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\cdot n!} = \frac{1}{3} - \frac{1}{2!5} + \frac{1}{3!7} - \frac{1}{4!9} + \cdots$$

③ Taylor series of $\sin(x)$ centered at $x = \frac{\pi}{2}$.

<u>Soln 1</u>: • Compute $\underset{\sin}{\sin\frac{\pi}{2}}$, $\underset{\sin'}{\cos\left(\frac{\pi}{2}\right)}$, $\underset{\sin''}{-\sin\left(\frac{\pi}{2}\right)}$, $\underset{\sin'''}{-\cos\left(\frac{\pi}{2}\right)}$, etc.

• Check ROC $= \infty$ & $|R_n(x)| \xrightarrow[n\to\infty]{} 0$ for any fixed x.



<u>Soln 2</u>: Write $\sin(x) = \cos\left(x - \frac{\pi}{2}\right)$

$$\sin(x) = \cos(\underbrace{x - \tfrac{\pi}{2}}_{z}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(x - \frac{\pi}{2}\right)^{2n}$$

• series for $\cos z$

• Substitute $z = x - \frac{\pi}{2}$

§3 <u>Proof of Lagrange Remainder formula</u>:

Pick $c=0$ as the center & write $R_N^{(f)}(x) = Q_N(x)\,(x-0)^{N+1}$ for $x \neq 0$

Want to show $Q_N(x) = \dfrac{f^{(N+1)}(b)}{(N+1)!}$ is independent of x.

Now, fix $x \neq 0$ & define a <u>new</u> function $F(t)$. for <u>all</u> t between 0 & x

How? Use

$$f(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \cdots + \frac{f^{(N)}(0)}{N!}(x-0)^N + Q_N(x)(x-0)^{N+1}$$

Replace every occurence of 0 by t & change signs:

$$F(t) = f(x) - f(t) - f'(t)(x-t) - \frac{f''(t)}{2!}(x-t)^2 - \cdots - \frac{f^{(N)}(t)}{N!}(x-t)^N - Q_N(x)(x-t)^{N+1}$$

• $F(0) = f(x) - f(0) - f'(0)\,x - \dfrac{f''(0)}{2!}x^2 - \cdots - \dfrac{f^{(N)}(0)}{N!}x^N - Q_N(x)\,(x)^{N+1} = 0$

• $F(x) = f(x) - f(x) = 0$

• F is cont on $[0,x]$ because $f(t), f'(t), \ldots, f^{(N)}(t)$ & $(x-t)^k$ are <u>all</u> continuous ($f^{(m)}$ is differentiable for <u>any</u> $m \geq 0$ so it's continuous!)

• F is differentiable on $(0,x)$ for the same reasons.

By the <u>Mean Value Thm</u>, we can find b between 0 & x with $F'(b) = 0$

For $\underline{N=1}$ $F'(t) = -\cancel{f'(t)} - f''(t)(x-t) + \cancel{f'(t)} + Q_N(x)\, 2(x-t) = -f''(t)(x-t) + 2Q_{N(x)}(x-t)$

So $F'(b) = -f''(b)(x-b) + 2Q_N(x)(x-b)$

$$0 = (x-b)\left(-f''(b) + 2Q_N(x)\right)$$

Since $b \neq x$, this forces $-f''(b) + 2Q_N(x) = 0$ so $Q_N(x) = \frac{f''(b)}{2}$.

$\underline{\text{For } N=2}$: $F'(t) = -\cancel{f'(t)} - \cancel{f''(t)(x-t)} + \cancel{f'(t)} - \frac{f'''(t)}{2!}(x-t)^2 + \cancel{\frac{2f''(t)}{2!}(x-t)}$

$$+ Q_N(x)\, 3(x-t)^2 = (x-t)^2\left(-\frac{f'''(t)}{2!} + 3Q_N(x)\right)$$

So $F'(b) = 0 = \underbrace{(x-b)^2}_{\neq 0}\left(-\frac{f'''(b)}{2!} + 3Q_N(x)\right)$

forces $\frac{f'''(b)}{3!} = Q_N(x)$

$\underline{\text{For general } N}$, we'll get similar cancellations that give a simple formula for $F'(t)$

$$F'(t) = -\frac{f^{(N+1)}(t)}{N!}(x-t)^N + (1+N)Q_N(x)(x-t)^N = (x-t)^N\left(-\frac{f^{(N+1)}(t)}{N!} + (N+1)Q_{N(x)}\right)$$

So $F'(b) = 0 = \underbrace{(x-b)^N}_{\neq 0}\left(-\frac{f^{(N+1)}(b)}{N!} + (N+1)Q_N(x)\right)$

so $0 = -\frac{f^{(N+1)}(b)}{N!} + (N+1)Q_N(x)$ forces $Q_N(x) = \frac{f^{(N+1)}(b)}{(N+1)!}$.

$\underline{\text{Conclusion}}$: $R_N(f)(x) = \frac{f^{(N+1)}(b)}{(N+1)!}(x-0)^{N+1}$.

- The formula for an arbitrary center c following by changing variables

write $z = x - c$ center c (for x) becomes center 0 (for z)

$f(x) = g(z+c)$ $\qquad R_N(g)(z) = \frac{g^{(N+1)}(\tilde{b})}{(N+1)!} z^{N+1}$ $\quad \tilde{b}$ in between 0 & z

and $f^{(n)}(x) = g^{(n)}(z+c)$ for all n.

$$= \frac{f^{(N+1)}(\tilde{b}+c)}{(N+1)!}(x-c)^{N+1} \quad \boxed{b := \tilde{b}+c} \text{ in between } c \text{ \& } z+c=x.$$

- $R_N(f)(x) = R_N(g)(z)$ by construction.