Lecture LVII: §14.5 Computations using Taylor's formula
§14.6 Applications to differential equations

Recall: $f(x) = f(c) + f'(c)(x-c) + \cdots + \frac{f^{(N)}(c)}{N!}(x-c)^N + R_N(f)(x)$

$$R_N(f)(x) = \frac{f^{(N+1)}(b)}{(N+1)!}(x-c)^{N+1} \quad \text{for some } b \text{ between } c \;\&\; x.$$

GOAL: Use $R_N(f)(x)$ to approximate values of $f(x)$ for x near c.

§1 Exponentials:

Want to estimate e up to m decimal places:

Write $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x so $e = e^1 = \sum_{n=0}^{\infty} \frac{1}{n!}$

Q: How far do we need to go to get $e = \sum_{n=0}^{N} \frac{1}{n!}$ up to m decimal places?

Soln: • Need to estimate $e - \sum_{n=0}^{N} \frac{1}{n!} = R_N(1)$

• Show $\boxed{|R_N(1)| < 0.5\ 10^{-m}}$ = Level of accuracy.

Ex: $m=1$ gives an error of at most 0.05 ⇝ "1st dec places agree"

$m=2$ ——— 0.005 ⇝ "1st two dec. ———"

$$|R_N(1)| = \left| f^{(N+1)}(b)\, \frac{1^{N+1}}{(N+1)!} \right| = \frac{e^b}{(N+1)!} \quad \text{for some } 0<b<1$$

But $e^b < e < 3$ so $|R_N(1)| < \frac{3}{(N+1)!}$

Conclusion: If we pick N so that $\frac{3}{(N+1)!} < 0.5\ 10^{-m}$, then $|R_N(1)| < 0.5\ 10^{-m}$

$$\boxed{10^m < \frac{(N+1)!}{6}}$$

TABLE:

N	$\frac{(N+1)!}{6}$
1	1/3
2	1
3	4
4	20
5	120
6	840
7	6,720
8	60,480

For $m=1$: $N=4$ works $e \sim 2+\frac{1}{2}+\frac{1}{6}+\frac{1}{24} = \frac{65}{24} = 2.7083$

$m=2$ $N=5$ works $e \sim 2.7106$

$m=3$ $N=7$ works $e \sim 2.7182539683$

$m=4$ $N=8$ works $e \sim 2.7182787699$

In general: $e \sim 2 + \frac{1}{2!} + \cdots + \frac{1}{N!}$ for each N gives $m = \lfloor \log_{10}\left(\frac{(N+1)!}{6}\right) \rfloor$ accuracy.

§2. Sines & Cosines: Same idea: Want $|R_N(x)| \leq 0.5\ 10^{-m}$ (given m, find N)

Ex 1 Approximate $\cos 93°$ to 6 decimal places.

Use Taylor series at $\frac{\pi}{2} = 90°$ & set $x = \frac{31}{60}\pi$ $(= 93°)$

Last time: $\sin(x) = 1 - \frac{1}{2!}\left(x-\frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x-\frac{\pi}{2}\right)^4 - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(x-\frac{\pi}{2}\right)^{2n}$

Term-by-term differentiation gives:

$$f_{(x)} = \cos(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!}\left(x-\frac{\pi}{2}\right)^{2n-1} = -\left(x-\frac{\pi}{2}\right) + \frac{1}{3!}\left(x-\frac{\pi}{2}\right)^3 - \cdots\cdots$$

$$|R_N^{(f)}(x)| = \frac{|f^{(N+1)}(b)|}{(N+1)!}\left|x-\frac{\pi}{2}\right|^{N+1} \leq \frac{\left|x-\frac{\pi}{2}\right|^{N+1}}{(N+1)!} \qquad (f^{(N+1)} = \pm\cos, \pm\sin)$$

$$\text{So } \left|R_N(f)\left(\tfrac{31}{60}\pi\right)\right| \leq \frac{\left(\frac{\pi}{60}\right)^{N+1}}{(N+1)!} \underset{\boxed{\text{WANT}}}{\leq} 0.5\ 10^{-6}$$

So $\underline{N=3}$ works!

Consequence: $\cos 93° = -\left(\frac{\pi}{60}\right) + \frac{1}{6}\left(\frac{\pi}{60}\right)^3 \approx \boxed{-0.052336}$

§3 Application to differential equations:

Input: A diff'l equation

Output: Power series solving the equation (with ROC > 0, ideally)

- Steps:
  1. Propose a solution $\sum_{n=0}^{\infty} a_n x^n$ with ROC > 0
  2. Differentiate term-by-term to find a recursive relation among a_n's.
  3. Write down the series and check if ROC > 0. If not, OUTPUT = "No power series sol'n"

Note: If we are lucky, we can recognize the power series as an elementary function.

Ex ①: $y'_{(x)} = y_{(x)}$ $\rightsquigarrow$ 1. $y(x) = \sum_{n=0}^{\infty} a_n x^n$

$$2.\quad y'(x) = \sum_{n=1}^{\infty} a_n n x^{n-1} \overset{m=n-1}{=} \sum_{m=0}^{\infty} a_{m+1}(m+1)x^m.$$

$$\overset{?}{\parallel}$$

$$y(x) = \sum_{m=0}^{\infty} a_m x^m$$

Equate coeff by coeff.

$$a_1 + 2a_2 x + 3a_3 x^2 + \cdots \quad = \quad a_0 + a_1 x + a_2 x^2 + \cdots$$

So $a_1 = a_0$

$2a_2 = a_1 \rightsquigarrow a_2 = \frac{a_1}{2} = \frac{a_0}{2}$

$3a_3 = a_2 \rightsquigarrow a_3 = \frac{a_2}{3} = \frac{a_0}{3!}$

$4a_4 = a_3 \rightsquigarrow a_4 = \frac{a_3}{4} = \frac{a_3}{4!}$

in general $a_n = \frac{a_0}{n!}$ $(0! = 1)$

3. Propose $y(x) = a_0 + a_0 x + \frac{a_0}{2} x^2 + \frac{a_0}{3!} x^3 + \cdots = a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Check ROC $= +\infty$ by Ratio Test $\lim_{n\to\infty} |x| \underbrace{\frac{n!}{(n+1)!}}_{=\frac{1}{n+1}} = 0$

We recognize this as the series for e^x.

General soln : $y(x) = a_0 e^x$ for any a_0 in $\mathbb{R}$.

Ex ② $y''(x) + y(x) = 0$ (Simple Harmonic Motion!)

1. $y(x) = \sum_{n=0}^{\infty} a_n x^n$

2. $y'(x) = \sum_{n=1}^{\infty} a_n n x^{n-1}$ & $y''(x) = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} = \sum_{m=0} a_{m+2} (m+1)(m+2) x^m$ $\quad m = n-2$

So

$$y''(x) = a_2 2 + a_3 3.2 x + a_4 4.3 x^2 + a_5 .5.4 x^3 + a_6 6.5 x^4 + \cdots$$

$\overset{?}{\|}$

$$-y(x) = -a_0 - a_1 x - a_2 x^2 - a_3 x^3 - a_4 x^4 - \cdots$$

gives $2a_2 = -a_0 \rightsquigarrow a_2 = -a_0/2$

$a_3 3! = -a_1 \rightsquigarrow a_3 = -\frac{a_1}{3!}$

$a_4 4.3 = -a_2 \rightsquigarrow a_4 = \frac{a_0}{4.3.2} = \frac{a_0}{4!}$

$a_5 5.4 = -a_3 \rightsquigarrow a_5 = \frac{a_1}{5!}$

$a_6 6.5 = -a_4 \rightsquigarrow a_6 = \frac{-a_0}{6!}$

Values of a_0 determine $a_2, a_4, a_6, \ldots$

Values of a_1 ——— $a_3, a_5, a_7, \ldots$

Even coeffs: $a_{2n} = (-1)^n \frac{a_0}{(2n)!}$, Odd coeffs: $a_{2n+1} = (-1)^n \frac{a_1}{(2n+1)!}$

We get 2 series $y_1(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ $\rightsquigarrow$ ROC $= \infty$

$y_2(x) = a_1 \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ $\rightsquigarrow$ ROC $= \infty$

So $y_1(x)$ & $y_2(x)$ converge <u>absolutely</u> & so will $y_1(x) + y_2(x)$

But $y_1(x) + y_2(x) = a_0\left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$

& we want to <u>mix</u> the even & odd powers. <u>NOTE</u> : ROC $= \infty$

We <u>can</u> rearrange the series because the series is absolutely convergent

So $y(x) = a_0 \underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}}_{= \cos(x)} + a_1 \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}}_{= \sin(x)}$

So $y(x) = a_0 \cos(x) + a_1 \sin(x)$.

<u>Ex</u> ③ Bessel's Eqn $\quad xy'' + y' + xy = 0$

1. $y = \sum_{n=0}^{\infty} a_n x^n$

2. $y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$, $\quad y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$

$= \sum_{m=0}^{\infty} a_{m+1}(m+1) x^m \qquad = \sum_{m=0}^{\infty} a_{m+2}(m+2)(m+1) x^m$

$\rightsquigarrow xy'' + y' + xy = \sum_{m=0}^{\infty} a_{m+2}(m+2)(m+1) x^{m+1} + \sum_{m=0}^{\infty} a_{m+1}(m+1) x^m + \sum_{n=0}^{\infty} a_n x^{n+1}$

$= \sum_{n=1}^{\infty} a_{n+1}(n+1) n x^n + \sum_{m=0}^{\infty} a_{m+1}(m+1) x^m + \sum_{n=1}^{\infty} a_{n-1} x^n =$

$n = m+1 \qquad\qquad n = m+1$

$= a_1 + \sum_{n=1}^{\infty} \left(a_{n+1}(n+1)n + a_{n+1}(n+1) + a_{n-1}\right) x^n \stackrel{?}{=} 0$

gives $a_1 = 0$, $\quad a_{n+1}(n+1)^2 + a_{n-1} = 0$

$a_{n+1} = \frac{-a_{n-1}}{(n+1)^2}$ for $n \geq 1$

$$\boxed{a_{m+2} = \frac{-a_m}{(m+2)^2} \quad \text{for } m \geq 0}$$

Conclusion: · ODD coeffs all 0

· Even coeffs $a_{2m} = \dfrac{(-1)^m a_0}{(2^2 4^2 \cdots (2m))^2} = \dfrac{(-1)^m a_0}{2^{2m}(m!)^2}$

(Check $a_2 = \dfrac{-a_0}{2^2}$, $a_4 = \dfrac{-a_2}{(4)^2} = \dfrac{+a_0}{2^2 \cdot 4^2}$

$y(x) = a_0 \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{2^{2n}(n!)^2}$

Name = Bessel function of order 0

(Notation: $J_0(x)$)

Check ROC $= +\infty$ by Ratio Test.