Lecture LVIII: §14.7 Operations on power series
§A16 Division of power series

GOAL: Use algebraic manipulations to compute Taylor series with ROC > 0 of function f without explicitly computing all $f^{(n)}(c)$. In particular:

1. Substitution
2. Product
3. Long Division

Key: If f can be expanded as a power series around a center c, this series MUST be the Taylor series of f around c. [Uniqueness!]

Ex ①: Substitution of one series in other one ($f(g(x))$)

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + \cdots \quad \text{for } |x|<1 = \text{ROC}$$

Q1: Series for $\frac{1}{1-x^4}$?

A: $g(x) = x^4$ & need $|x^4| <$ ROC of f.

So $f(x^4) = \frac{1}{1-x^4} = 1 + (x^4) + (x^4)^2 + \cdots = \sum_{n=0}^{\infty} (x^4)^n = \sum_{n=0}^{\infty} x^{4n}$.

with ROC = 1

Q2 Series for $\frac{x^5}{1-x^4}$? A $x^5 \sum_{n=0}^{\infty} x^{4n} = \sum_{n=0}^{\infty} x^{4n+5}$ ROC = 1 also.

Conclusion: $h(x) = \frac{1}{1-x^4}$, $P(x) = \frac{x^5}{1-x^4}$

$$h^{(n)}(0) = \begin{cases} 0 & \text{if } n \text{ not divisible by } 4 \\ n! & \text{else} \end{cases} \qquad P^{(n)}(0) = \begin{cases} 0 & \text{if } n \neq 4k+5 \text{ (for some } k) \\ n! & \text{else} \end{cases}$$

Substitution Rule:

$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ & $g(x) = b_0 + b_1 x + b_2 x^2 + \cdots$

$$\rightsquigarrow f(g(x)) = a_0 + a_1(b_0 + b_1 x + b_2 x^2 + \cdots) + \underbrace{a_2(b_0 + b_1 x + \cdots)^2}_{a_2\, b_0^2 + 2b_0 b_1 x + (b_0 b_2 + b_1^2 + b_2 b_0)x^2 + \cdots} + \cdots$$

If ROC(f) = R > 0 & ROC(g) = $\tilde{R}$ > 0, then $f(g(x))$ has a power series expansion whenever $|x| < \tilde{R}$ & $|g(x)| < R$ (Need: $|b_0| < R$)

⚠ Calculation involves powers of series ⇝ Product of series = ??

Ex ② Product of 2 power series.

$$f(x) = e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad ; \qquad g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

ROC $= \infty$ ROC $= \infty$

$f(x)\,g(x)$ is a power series with ROC $= \min\{\text{ROC}(f), \text{ROC}(g)\} = \infty$

Q: How? A: Use distribution laws and collect coefficients for each power of x.

$$\begin{array}{rlr}
f(x)g(x) = & x \; \boxed{-\frac{x^3}{3!}} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots & 1\cdot g(x) \\
& + \boxed{x^2} \; \boxed{-\frac{x^4}{3!}} + \frac{x^6}{6!} - \frac{x^8}{7!} + \cdots & + \; x\cdot g(x) \\
& + \frac{x^3}{2} - \frac{x^5}{12} + \frac{x^7}{2!\,5!} - \frac{x^9}{2.7!} + \cdots & + \; \frac{x^2}{2}\, g(x) \\
& + \boxed{\frac{x^4}{3!}} - \frac{x^6}{3!\,3!} + \frac{x^8}{3!\,5!} - \frac{x^{10}}{3!\,7!} + \cdots & \frac{x^3}{3!}\, g(x) \\
& \vdots & \vdots
\end{array}$$

coeff of $x = 1$ coeff of x^3: $-\frac{1}{3!} + \frac{1}{2} = \frac{1}{3}$ .... (complicated formulas)

" " $x^2 = 1$ coeff of x^4: $-\frac{1}{3!} + \frac{1}{3!} = 0$

Another example: $f(x) = \ln(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right)$ ROC $= 1$

$g(x) = \frac{1}{x-1} = -(1 + x + x^2 + x^3 + \cdots)$ ROC $= 1$

$$\left.\begin{array}{rll}
f(x)g(x) = & x + x^2 + x^3 + x^4 + \cdots & = 1\, g(x) \\
+ & \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{2} + \cdots & = x\, g(x) \\
& \frac{x^3}{3} + \frac{x^4}{3} + \cdots & = \frac{x^2}{2}\, g(x) \\
& \ddots & \vdots \\ \hline
& \multicolumn{2}{l}{x + \left(1+\frac{1}{2}\right)x^2 + \left(1+\frac{1}{2}+\frac{1}{3}\right)x^3 + \left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)x^4 + \cdots}
\end{array}\right\} = \sum_{n=0}^{\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) x^n$$

Q: When does this method work? A: We are rearranging a series! Need $f(x)$ & $g(x)$ to be absolutely convergent, so need to work with $|x| < \text{ROC}(f)$ & $|x| < \text{ROC}(g)$

Conclusion:
- ROC $= \min\{\text{ROC}(f), \text{ROC}(g)\} > 0$.
- Power series expansions for f & g MUST have the same center.

Formally:

Product Rule

$f(x)=\sum_{n=0}^{\infty} a_n x^n$ with ROC $=R_1>0$; $g(x)=\sum_{n=0}^{\infty} b_n x^n$ with ROC $=R_2>0$

Term-by-Term multiplication & add along columns:

$$
\begin{array}{rlllll}
 & a_0 g(x) = & a_0b_0 + & a_0b_1x & + a_0b_2x^2 + & a_0b_3x^3 + \cdots \\
+ & a_1 x g(x) = & & a_1b_0x & + a_1b_1x^2 + & a_1b_2x^3 + \cdots \\
 & a_2 x^2 g(x) = & & & a_2b_0x^2 + & a_2b_1x^3 + \cdots \\
 & \vdots & & & & \\
\hline
 & & a_0b_0 & + (a_0b_1+a_1b_0)x & + (a_0b_2+a_1b_1+a_2b_0)x^2 & + \cdots
\end{array}
$$

NO terms below staircase

Proposition: $\boxed{f(x)\,g(x)=\sum_{n=0}^{\infty}\underbrace{\left(\sum_{k=0}^{n} a_k b_{n-k}\right)}_{\text{coeffs of } fg} x^n}$ (*) & the series converges absolutely if $|x|<R=\min\{R_1,R_2\}$

Why? Take partial sums of $f(x)$ & $g(x)$:

$s_n = a_0 + a_1x + \cdots + a_nx^n$
$t_n = b_0 + b_1x + \cdots + b_nx^n$ $\leadsto s_nt_n=\sum_{p=0}^{2n}\left(\sum_{k=0}^{p} a_k b_{p-k}\right)x^p$.

Rearrange s_nt_n as:

$$
\begin{array}{c|cccc}
x^0 & a_0b_0 & a_0b_1 & a_0b_2 & a_0b_3 \\
x^1 & a_1b_0 & a_1b_1 & a_1b_2 & a_1b_3 \;\cdots \\
x^2 & a_2b_0 & a_2b_1 & a_2b_2 & a_2b_3 \;\cdots \\
x^3 & a_3b_0 & a_3b_1 & a_3b_3 & a_3b_3 \\
x^4 & \vdots & \vdots & &
\end{array}
$$

rows: $a_i x^i g(x)$
cols: $b_j x^j f(x)$

① Sum the Ls gives s_nt_n
② Sum along antidiagonal is partial sum of (RHS) in (*)

By absolute convergence, we can rearrange the series in ANY way & get the same sum.

- Way ①: $s_nt_n \xrightarrow[n\to\infty]{} f(x)\,g(x)$
- Way ② series in (RHS) in (*)

Remark: Really need abs convergence for (*) to work.

Ex: $f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{\sqrt{n+1}} = 1-\frac{x}{\sqrt{2}}+\frac{x^2}{\sqrt{3}}-\cdots$ not abs conv for $x=1$, only cond. conv. ROC $=1$.

Series for $f^2(x)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{(-1)^k}{\sqrt{k+1}}\,\frac{(-1)^{n-k}}{\sqrt{n-k+1}}\right)x^n=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{1}{\sqrt{(k+1)(n-k+1)}}\right)x^n$

This series diverges for $x=1$ but $f(x)^2$ is defined at $x=1$.

Ex ③ Long division of power series $\frac{f(x)}{g(x)}$ when $g(0)=1$ (any $g(0)\neq 0$ works) gives a new power series with ROC > 0 (need to avoid zeros of $g(x)$!)

Ex: $\tan(x) = \frac{\sin x}{\cos x}$ will have a power series expansion with ROC $= \frac{\pi}{2}$ (even though ROC for sin & cos is ∞)

($\cos\left(\frac{\pi}{2}\right) = \cos\left(\frac{-\pi}{2}\right) = 0$ & no zeroes in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$)

$$x + \boxed{\tfrac{1}{3}x^3} + \tfrac{2}{15}x^5 + \cdots = \tan(x)$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots \quad \Big) \; x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sin(x)$$

$$- \quad x - \frac{x^3}{2} + \frac{x^5}{24} - \cdots$$

$$\boxed{\tfrac{1}{3}x^3} - \frac{1}{30}x^5 + \cdots$$

$$- \quad \frac{1}{3}x^3 - \frac{1}{6}x^5 + \cdots$$

$$\frac{2}{15}x^5 + \cdots$$

Key $\cos x = \overset{\neq 0}{1} +$ terms with x

$\rightsquigarrow$ We can write $\frac{1}{\cos x}$ as a power series (Appendix A16)

Prop If $\sum_{n=0}^{\infty} b_n x^n$ has $b_0 \neq 0$ & ROC > 0, then $\frac{1}{\sum_{n=0}^{\infty} b_n x^n}$ has a power series expansion with ROC > 0

Why? (1) Propose $\frac{1}{\sum_{n=0}^{\infty} b_n x^n} = \sum_{n=0}^{\infty} c_n x^n$ & find c_n by recursion

$$1 = \left(\sum_{n=0}^{\infty} b_n x^n\right)\left(\sum_{n=0}^{\infty} c_n x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} b_{n-k} c_k\right) x^n$$

$$= b_0 c_0 + (b_0 c_1 + b_1 c_0) + (b_0 c_2 + b_1 c_1 + b_2 c_0) x^2 + \cdots$$

$\rightsquigarrow$ $b_0 c_0 = 1$ & $b_0 \neq 0$ so $\boxed{c_0 = \frac{1}{b_0}}$

$b_0 c_1 + b_1 c_0 = 0$ & $b_0 \neq 0$ $\rightsquigarrow$ $c_1 = \frac{-b_1}{b_0} c_0 = \boxed{\frac{-b_1}{b_0^2}}$

$b_0 c_2 + b_1 c_1 + b_2 c_0 = 0$ & $b_0 \neq 0$ $\rightsquigarrow$ $c_2 = -\frac{1}{b_0}\big(\underbrace{b_1 c_1 + b_2 c_0}_{\text{known values!}}\big)$

(2) Show the series $\sum_{n=0}^{\infty} c_n x^n$ has a positive ROC

$$\rightarrow c_n = -\sum_{k=0}^{n-1} \frac{b_{n-k} c_k}{b_0} \quad \text{for all } n$$

We can assume $b_0 = 1$ (& so $c_0 = 1$). Otherwise, $1 = b_0 \left(\sum_{n=0} \frac{b_n}{b_0} x^n\right)\left(\frac{1}{b_0}\sum_{n=0}^{\infty} (c_n b_0) x^n\right)$

constant coeff $= 1$

• Since $\sum_{n=0}^{\infty} b_n x^n$ has ROC $= R > 0$, pick $0 < r < R$ & set

$\sum_{n=0}^{\infty} |b_n| r^n$ converges so $|b_n| r^n \xrightarrow[n\to\infty]{} 0$

In particular, the sequence $\{|b_n| r^n\}_n$ is bounded & we can find $\boxed{K \geq 1}$ (because $b_0 = 1$)

with $|b_n| r^n \leq K$ for all n

$$\boxed{|b_n| \leq \frac{K}{r^n}}$$

• $|c_0| = 1 \leq K$

• $|c_1| = |b_1 c_0| = |b_1| \leq \frac{K}{r}$

• $|c_2| = |b_1 c_1 + b_2 \underset{1}{c_0}| \leq |b_1 c_1| + |b_2| \leq \frac{K}{r}\frac{K}{r} + \frac{K}{r^2} K = 2\frac{K^2}{r^2}$

• $|c_3| = |b_1 c_2 + b_2 \overset{1}{c_1} + b_3| \leq |b_1 c_2| + |b_2 c_1| + |b_3| \leq \frac{K}{r}\frac{2K^2}{r^2} + \frac{K}{r^2}\frac{K}{r} + \frac{K}{r^3}$

$\underset{K\geq 1}{\leq} \frac{2K^3}{r^3} + \frac{K^3}{r^3} + \frac{K^3}{r^3} = \frac{4K^3}{r^3}$

So $|c_3| \leq \frac{2^2 K^3}{r^3}$ $\rightsquigarrow$ $\boxed{\text{Propose } |c_\ell| \leq \frac{2^{\ell-1} K^\ell}{r^\ell} \text{ for all } \ell}$

In general:

$|c_n| \leq |b_1 c_{n-1}| + |b_2 c_{n-2}| + \cdots + |b_{n-1} c_1| + |b_n|$

$\leq \frac{K}{r} \frac{2^{n-2} K^{n-1}}{r^{n-1}} + \frac{K}{r^2} \frac{2^{n-3} K^{n-2}}{r^{n-2}} + \cdots + \frac{K}{r^{n-1}} \frac{K}{r} + \frac{K}{r^n}$

$\underset{1 \leq K \leq K^2 \leq \cdots \leq K^n}{\leq} (\underbrace{2^{n-2} + 2^{n-3} + \cdots + 2^2 + 2 + 1}_{\frac{2^{n-1}-1}{2-1}} + 1) \frac{K^n}{r^n} = \boxed{2^{n-1}\frac{K^n}{r^n}}$

So $\sum_{n=0}^{\infty} |c_n||x|^n \leq \sum_{n=0}^{\infty} 2^{n-1} \frac{K^n}{r^n}|x|^n \leq \frac{1}{2}\sum_{n=0}^{\infty} \left(\frac{2K|x|}{r}\right)^n$ & this converges if

$\left|\frac{2Kx}{r}\right| < 1$ that is $|x| < \frac{r}{2K}$. This is true for all $0 < r < R$

Conclusion: The ROC of the series $\sum_{n=0}^{\infty} c_n x^n$ is at least $\frac{R}{2K} > 0$.