Lecture LIX: §14.3 (cont.) Differentiation & Integration of power series
A15: Uniform convergence of power series

§1. Derivatives & Integrals of power series:

Theorem: Fix $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with ROC $= R > 0$. Then:

① f is cont on $(-R, R)$

② f is differentiable term-by-term on $(-R,R)$ & $f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots = \sum_{n=1}^{\infty} a_n n x^{n-1}$

③ f is integrable ———— on $(-R,R)$ & $\int f(x)\,dx = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1} + C$.

In particular the 2 power series have radius of convergence $= R$.

Why is this true? Let's start with ② & ③ & show ROC $\geq R$ in both cases.

• First attempt at ②: Assume we know ROC $= R$ for $f(x)$ by the Root Test:

$$|x| \lim_{n\to\infty} \sqrt[n]{|a_n|} < 1 \quad \text{gives convergence, so } R = \frac{1}{\lim_{n\to\infty} \sqrt[n]{|a_n|}}$$

(This is close to always being true: $\frac{1}{R} = \limsup \sqrt[n]{|a_n|}$)

$$\text{Then } \lim_{n\to\infty} \sqrt[n]{|\underbrace{n a_n x^{n-1}}_{n^{\text{th}} \text{ term of series in ②}}|} = \lim_{n\to\infty} \underbrace{\sqrt[n]{|n|}}_{\to 1}\ \underbrace{\sqrt[n]{|a_n|}}_{\to 1/R}\ \underbrace{|x|^{\frac{n-1}{n}}}_{\to |x|} = \frac{|x|}{R} \quad \text{exists}$$

The Root Test says: $\begin{cases} \text{converge if } \frac{|x|}{R} < 1 \\ \text{diverge " } \frac{|x|}{R} > 1 \end{cases}$ so ROC for ② is R.

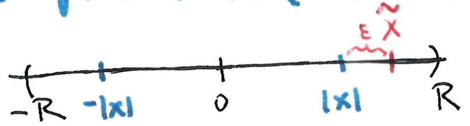
~~Q: What is converging?~~ • [illegible] okay this [illegible]

Consequence: Same argument repeated over & over again shows f is diff'ble up to any order (once we know ② represents f').

• Second attempt at ②: Let's show ② without the Root Test for f.

• Know $\sum_{n=0}^{\infty} |a_n x^n|$ converges if $|x| < R$

• Given x in $(-R, R)$, can find $\varepsilon > 0$ so that $\tilde{x} = |x| + \varepsilon$ lies in $(0, R)$

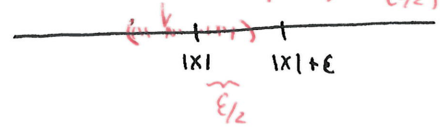


$\leadsto$ The series $\sum_{n=0}^{\infty} |a_n| \tilde{x}^n$ converges.

Claim: $|n x^{n-1}| \leq (\underbrace{|x| + \varepsilon}_{= \tilde{x}})^n$ for n large enough

Reason: $\sqrt[n]{|n|}\;|x|^{\frac{n-1}{n}} \xrightarrow[n\to\infty]{} |x| < |x|+\varepsilon$ so for n large enough

$\sqrt[n]{n}\;|x|^{\frac{n-1}{n}}$ for $n \geq n_0(\varepsilon/2)$



$$|n|\,x^{n-1} < (|x|+\varepsilon)^n.$$

Consequence: $|n a_n x^{n-1}| \leq |a_n|\,\tilde{x}^n$ for $n \geq n_0$

By Comparison $\sum_{n=n_0}^{\infty} |n a_n x^{n-1}| \leq \sum_{n=n_0}^{\infty} |a_n|\,\tilde{x}^n < \infty$ converges

Then $\sum_{n=0}^{\infty} |n a_n x^{n-1}| = |a_0| + 2|a_2|\,|x| + \cdots + |n_0-1|\,|a_{n_0-1}|\,|x|^{n_0-2} + \sum_{n=n_0}^{\infty} |n a_n x^{n-1}|$

converges. for $|x|<R$.

Q: What's missing? Still don't know why the series $= f'(x)$!!!

We'll need Absolute convergence for this..

• First attempt at ③: Rewrite $C + \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1} = \sum_{n=0}^{\infty} b_n x^n$ $\left[\begin{array}{l} b_0 = C,\ b_1 = a_0 \\ b_2 = \frac{a_1}{2}, \ \dots \end{array}\right]$

So $|b_{n+1}| = \left|\frac{a_n}{n+1}\right| \leq |a_n|$ for $n \geq 0$.

Since $\sum |a_n|\,|x|^n$ converges if $|x|<R$, by comparison $\sum_{n=0}^{\infty} |b_n|\,|x|^n$ also converges for $|x|<R$. Conclude: ROC for ③ is at least R.

Now: Derivation & Integration are "inverse operations", so ROC $\geq R$ forces $\boxed{ROC = R}$ (otherwise f would have ROC $= R' =$ ROC(③) $> R$, which we know is false!)

Q: What's missing? Still don't know why the series $= \int f(x)\,dx$!!!

As with ②, we'll need Absolute convergence.

§2. Uniform convergence of power series:

Recall. $f(x) = \sum_{n=0}^{\infty} a_n x^n = \underbrace{\sum_{n=0}^{N} a_n x^n}_{\substack{(N^{th}\text{ partial sum}) \\ = S_N(f)(x)}} + R_N(f)(x)$

Prop: If ROC of f is $R>0$, then $S_N(f)(x) \xrightarrow[N\to\infty]{} f(x)$ if and only if $R_N(f)(x) \xrightarrow[N\to\infty]{} 0$ for every fixed x.

In other words, given $\varepsilon>0$, we can find $\boxed{N_0 = N_0(x,\varepsilon) > 0}$ so that $|R_N(f)(x)| < \varepsilon$ for $N \geq N_0$

⚠ N_0 in principle depends both on ε & x!

Definition $\boxed{\text{Uniform convergence} = N_0 \text{ is independent of } x}$ (as long as x lies in $[-R', R']$ for $R' < R$.)

More precisely: Fix $R' < R$. Then, $R_N(x) \xrightarrow[n\to\infty]{} 0$ uniformly on $[-R', R']$, meaning $N_0 = N_0(\varepsilon, R')$ is independent of x.

Prop The series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-R', R']$ for every $R' < R$ ($= ROC(f)$)

Why? Since $|x| \leq R' < R$, we get $|R_N(x)| = \left|\sum_{k=N+1}^{\infty} a_k x^k\right| \leq \sum_{k=N+1}^{\infty} |a_k| \underbrace{|x|^k}_{\leq R'^k}$

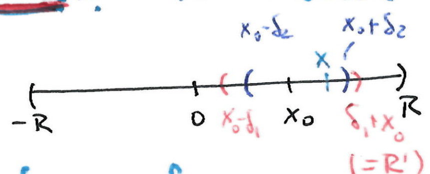
$\Rightarrow |R_N(x)| \leq \sum_{k=N+1}^{\infty} |a_k| R'^k = $ tail of $\sum_{n=0}^{\infty} |a_n| (R')^n$

Since $f(R')$ converges absolutely because $R' < R$, we can find $N_0 = N_0(\varepsilon, R')$ where

$$\sum_{k=N_0+1}^{\infty} |a_k| R'^k < \varepsilon \quad \text{so} \quad |R_N(x)| < \varepsilon \quad \text{if} \quad N \geq N_0.$$

Consequence 1: Power series are continuous (① in Thm)

Proof:



Given x_0 in $(-R, R)$ want to show

$\lim_{x \to x_0} f(x) = f(x_0)$.

Pick any $\varepsilon > 0$ & write $f(x) = \underbrace{S_N(x)}_{\text{poly}} + \underbrace{\boxed{R_N(x)}}_{\text{tail}}$ Q: What N to pick?

Pick $\delta_1 > 0$ with $-R < x_0 - \delta_1 < x_0 + \delta_1 < R$ ($\delta_1 = \frac{R - |x_0|}{2}$ will do)

Call $R' = \max\{|x_0 + \delta_1|, |x_0 - \delta_1|\} < R$ by construction.

By uniform convergence, we can find $N_0 = N_0(\varepsilon, R')$ so that $|R_N(y)| < \frac{\varepsilon}{3}$ for every y in $[-R', R']$ for every $N \geq N_0$.

Pick $N = N_0$ & write $f(y) = S_{N_0}(y) + R_{N_0}(y)$

Since x & x_0 lie in $[-R', R']$, we have $|R_{N_0}(x)| < \frac{\varepsilon}{3}$ & $|R_{N_0}(x_0)| < \frac{\varepsilon}{3}$

Now
$$\begin{aligned}|f(x) - f(x_0)| &= |S_{N_0}(x) + R_{N_0}(x) - (S_{N_0}(x_0) + R_{N_0}(x_0))| \\ &= |(S_{N_0}(x) - S_{N_0}(x_0)) + (R_{N_0}(x) - R_{N_0}(x_0))| \\ &\leq |S_{N_0}(x) - S_{N_0}(x_0)| + |R_{N_0}(x)| + |R_{N_0}(x_0)|\end{aligned}$$

• But $S_{N_0}(y)$ is a polynomial $\left(= \sum_{n=0}^{N_0} a_n y^n\right)$ so given x_0 we can find $\delta_2 > 0$

so $|S_{N_0}(x) - S_{N_0}(x_0)| < \frac{\varepsilon}{3}$ if $|x - x_0| < \delta_2$.

• Pick $\delta = \min\{\delta_1, \delta_2\} > 0$.

Then
$$|f(x) - f(x_0)| \le |S_{N_0}(x) - S_{N_0}(x_0)| + |R_{N_0}(x)| + |R_{N_0}(x_0)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

if $|x - x_0| < \delta$.

<u>Conclusion</u>. $\lim_{x \to x_0} f(x) = f(x_0)$.

Consequence 2: Term-by-Term integration

<u>Proof</u>: Assume $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has ROC $= R > 0$. Pick $-R < c < d < R$.

Want to show: $\int_c^d \sum_{n=0}^{\infty} a_n x^n \, dx = \sum_{n=0}^{\infty} \int_c^d a_n x^n \, dx = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1} \Big|_c^d$

<u>How</u>? Write $f(x) = \overset{\text{CONT}}{S_N(x)} + \overset{\text{CONT}}{R_N(x)}$ (Need to find suitable N.)

• $S_N(x)$ is a polynomial, so we integrate term-by-term $\int_c^d \sum_{n=0}^{N} a_n x^n \, dx = \sum_{n=0}^{N} \frac{a_n x^{n+1}}{n+1} \Big|_c^d$

• $\int_c^d f(x)\, dx = \int_c^d S_N(x)\, dx + \int_c^d R_N(x)\, dx$

$$= \sum_{n=0}^{N} \frac{a_n x^{n+1}}{n+1} \Big|_c^d + \int_c^d R_N(x)\, dx \overset{?}{=} \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1} \Big|_c^d$$

Need to show $\Big| \underbrace{\int_c^d R_N(x)\, dx}_{\text{this is a number for each } N} \Big|$ "=" $\Big| \text{tail of } \sum_{k=0}^{\infty} \int_c^d a_n x^n \, dx \Big| \xrightarrow[N \to \infty]{} 0$

Use Uniform convergence! Given $\varepsilon > 0$, want to find N_0 with $\left| \int_c^d R_N(x)\, dx \right| < \varepsilon$ if $N \ge N_0$.

Eg. 

$\leadsto R' = |c|$.

Pick $R' = \max\{|c|, |d|\} < R$. Pick N_0 so: $|R_N(x)| < \frac{\varepsilon}{d-c}$ if $N \ge N_0$ for any x in $[-R', R']$

Then $\left| \int_c^d R_N(x)\, dx \right| \le \int_c^d |R_N(x)|\, dx < \int_c^d \frac{\varepsilon}{d-c}\, dx = \frac{\varepsilon}{d-c} x \Big|_c^d = \varepsilon$ if $N \ge N_0$

<u>Conclusion</u>: $\int \sum_{n=0}^{\infty} a_n x^n \, dx = C + \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$.

Consequence 3: Term-by-Term differentiation

Proof: Only missing point: the series $g(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ with ROC $= R$ represents f'.

Since $g(x)$ is continuous on $(-R, R)$ & we can integrate Term-by-Term,

we get $\displaystyle\int_0^x \underbrace{\sum_{n=1}^{\infty} n a_n t^{n-1}}_{= g(t)} dt = \sum_{n=1}^{\infty} \int_0^x n a_n t^{n-1} dt = \sum_{n=1}^{\infty} a_n t^n \Big|_0^x = \sum_{n=1}^{\infty} a_n x^n = f(x) - a_0$

So $f(x) = a_0 + \displaystyle\int_0^x g(t)\, dt$ is diff'ble by FTC &

$f'(x) = g(x) = \displaystyle\sum_{n=1}^{\infty} n a_n x^{n-1}$ as we wanted to show!