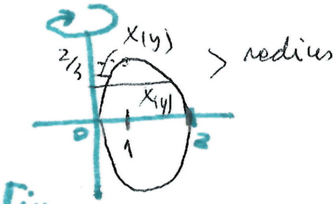
Problem: Revolve the loop about the y-axis $\quad 9y^2 = x(3-x)^2$

Find the surface area.



Soln: We draw the loop (it's symmetric about x-axis)

(x-intercepts: 0 & 3, values of x must be ≥ 0)

Find the top vertex by $y'=0$, using impl. differentiation

$$18yy' = (3-x)^2 - 2x(3-x) = (3-x)(3-x-2x) = (3-x)(3-3x)$$
$$= 3(3-x)(1-x)$$

So $y'=0$ when $\underline{x=1}$ $(y\neq 0)$. $\rightsquigarrow y = 2/3$

Area = 2 (Inner Area) + 2 (Out Area) $\qquad$ In both cases $0 \leq y \leq 2/3$

For $x \leq 1$ $\qquad$ for $x \geq 1$

Need to integrate $\quad 2\pi\, x(y)\sqrt{1+(x'(y))^2}\,dy$

We compute x' with implicit differentiation.

As above $\quad 18y = 3(3-x)(1-x)x' \quad$ so $\quad x' = \dfrac{6y}{(3-x)(1-x)}$

$$1+(x')^2 = \frac{(3-x)^2(1-x)^2+36y^2}{(3-x)^2(1-x)^2} \underset{\substack{\text{use eqn} \\ 9y^2=\dots}}{=} \frac{(3-x)^2(1-x)^2+4x(3-x)^2}{(3-x)^2(1-x)^2}$$

$$= \frac{(1-x)^2+4x}{(1-x)^2} = \frac{1+x^2-2x+4x}{(1-x)^2} = \frac{(1+x)^2}{(1-x)^2}$$

So $\quad 2\pi x(y)\sqrt{1+x'(y)}\,dy = 2\pi\, x(y)\dfrac{1+x(y)}{(1-x(y))}\,dy$

Now, we use $y = y(x)$ because we can solve eqn for y but NOT for x.

$$2\pi x\frac{1+x}{1-x}\,dy = 2\pi x\frac{1+x}{1-x}\,y'\,dx = 2\pi\frac{(1+x)}{1-x}x\,\frac{(3-x)(1-x)}{6y}\,dx$$

$$\underset{\text{use eqn.}}{=} \pi x\frac{(3-x)(1+x)}{\sqrt{x}(3-x)}\,dx = \pi\sqrt{x}(1+x)\,dx$$

$$\text{Area} = 2\int_0^3 \pi\underbrace{\sqrt{x}(1+x)}_{=x^{1/2}+x^{3/2}}\,dx = 2\pi\left(\frac{2}{3}x^{3/2}+\frac{2}{5}x^{5/2}\right)\Big|_0^3 = 2\pi\left(2\sqrt{3}+\frac{18\sqrt{3}}{5}\right) = \boxed{\frac{56}{5}\sqrt{3}\pi}$$