Lecture I : § 2.1 & 2.2 The problem of tangents, slopes

Textbook: G.F. Simmons, "Calculus with Analytic Geometry" (2^{nd} edition)

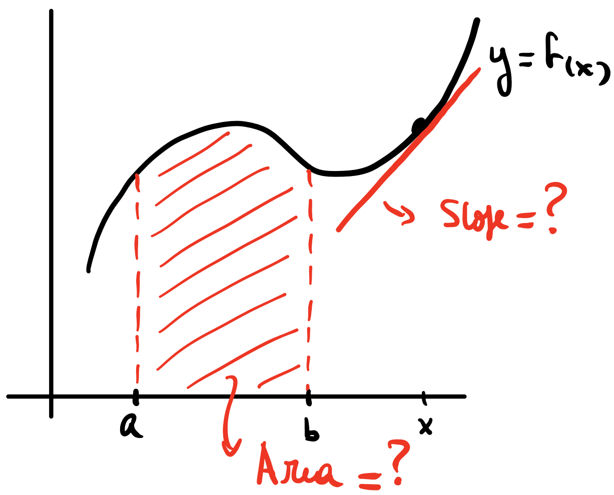
§ 2.1 What is Calculus?

Calculus = "To compute" (Today = The problem of tangents)

(1) **Two Fundamental Questions**:

- Q1: Find the rate at which a variable quantity is changing.
- Q2: Describe a varying quantity when its rate of change is known.
  ↳ "inverse problem"

• **Geometric Interpretations**:



- P1: Find tangent lines to curves.
  ($y = \boxed{m}\, x + b$ m = slope $y = f(x)$)
- P2: Find the area under a curve.

P1: Differential Calculus (Chapters 2–5)	⟷	P2: Integral Calculus (Chapters 6–12)

Fundamental Thm of Calculus §6.6

Part 3: Sequences & Infinite series.

(2) Why Calculus?

- From computations going back to Archimedes & the Greeks to its formalization by
  - Newton (1642 - 1727)
  - Leibniz (1646 - 1716)
- Part of the basic language of Science
- Used to describe continuous motions:

Examples: ① Motions of planets + gravity (eg discovery of Neptune)

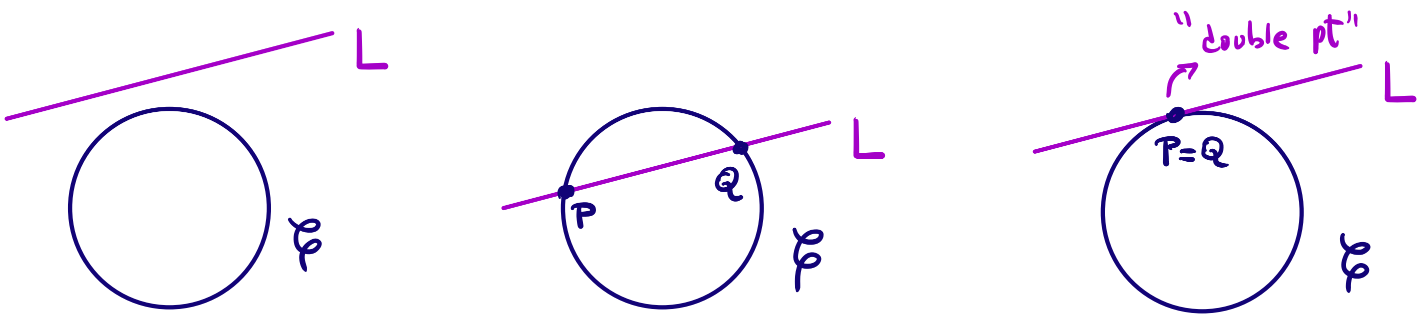
② Biology: Hodgkins-Huxley eqn describing the action potential across neurons in the brain.

③ Economics: Black-Scholes eqn modeling option pricing in financial markets.

- Why formalize & axiomatize? By doing this we can concentrate on the underlying structures of different phenomena. We gain flexibility by means of abstraction.

(3) What is a tangent line? "Tangible = to touch"

- IDEA 1: Look at lines relative to curves (example = circle $\mathcal{C}$)



- $\mathcal{C}$ on ONE side of L
- $\mathcal{C}$ & L don't meet

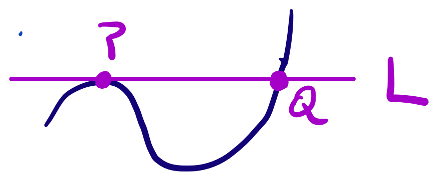
- $\mathcal{C}$ on TWO sides of L
- $\mathcal{C}$ & L meet at 2 pts (P & Q) (Secant Line)

- $\mathcal{C}$ on ONE side
- $\mathcal{C}$ & L meet at 1 pt (P) (Tangent Line)

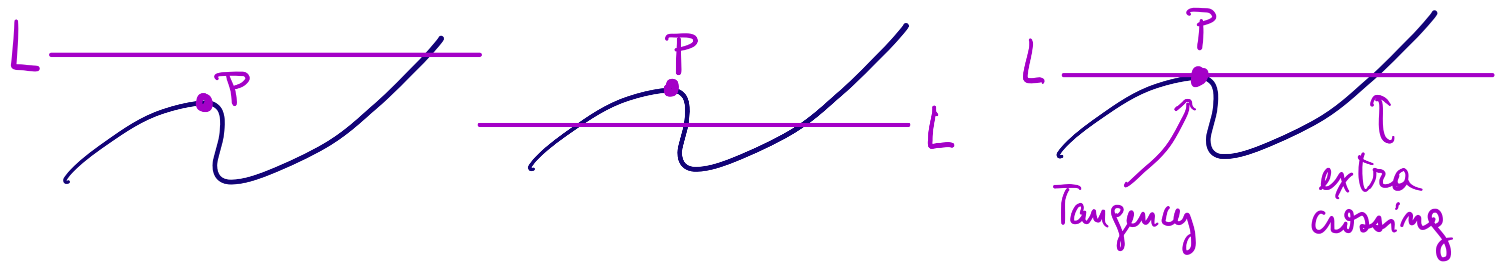
Conclude: A definition via "counting the number of intersection points" is too restrictive. It also fails for general curves

- IDEA 2: Think of the curve as being on "one side" of the line or on both & meeting at one point.

  This also fails in general

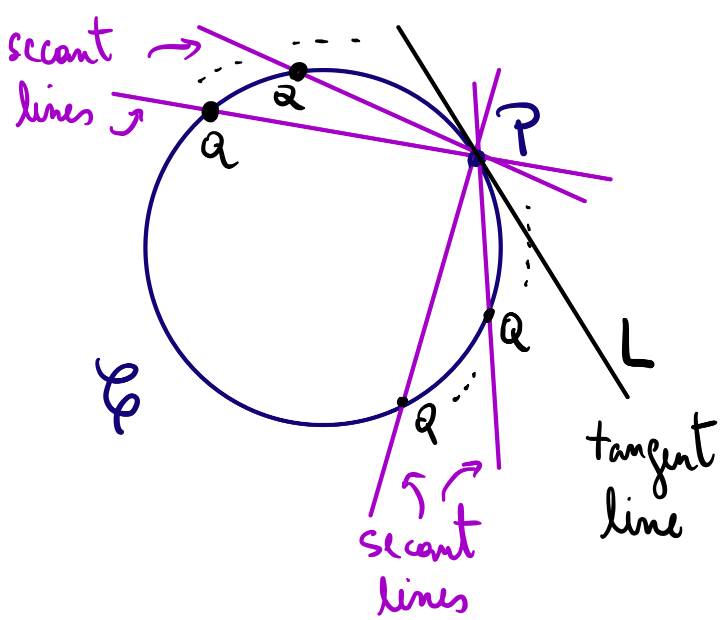


Solution: Combine these two ideas but in a local situation
(local = around the point of interest)



- Natural question: How to formalize our intuitive notion of a tangent line?
- Answer (Fermat ~ 1630): "Tangent lines are LIMITS of secant lines"

Geometric Description	Fermat's idea

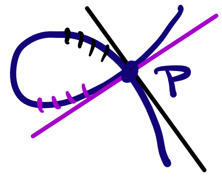


"Approach P from both sides"

(can use a sequence of secant lines)

1. Fix P & a second pt Q on the curve $\mathcal{C}$
2. Draw the (secant) line through P & Q
3. Move Q towards P & see how the secant line moves along with Q
4. Provided the tangent line L at P exists, the secant lines will approach L in the limit

Example Alpha curve $[\, Y^2 - X^3 - X^2 = 0 \,]$

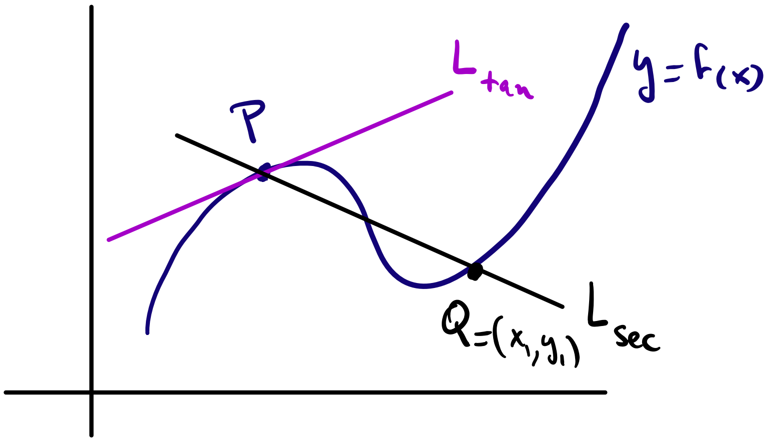


has no tangent line at P.

(two possible limits to a sequence of secant lines)

Remark: Almost all of Calculus involves some limiting process

§2.2 How to calculate the slope of a Tangent?



Recall: A non-vertical line through $P = (x_0, y_0)$ has equation

$$\boxed{Y = \underbrace{m}_{\text{slope}} (X - x_0) + y_0}$$

- The only parameter we need to determine is m.
- m is obtained via a limiting process ⇝ we need coordinates!
- Equation of L_{sec} $\quad Q = (x_1, y_1)$ satisfies

$$Y_1 = m_{sec}(x_1 - x_0) + y_0$$

⇝ We can solve for m_{sec} because $x_1 \neq x_0$ (L_{sec} is not vertical!)

$$m_{sec} = \frac{Y_1 - Y_0}{x_1 - x_0}$$

If $L_{Tan} = \lim\limits_{\substack{Q \to P \\ (Q \neq P)}} L_{sec}$, then $m_{Tan} = \lim\limits_{\substack{Q \to P \\ (Q \neq P)}} m_{sec}$

Conclude:
$$\boxed{m_{Tan} = \lim_{\substack{(x_1, y_1) \to (x_0, y_0) \\ (x_1, x_0) \neq (y_1, y_0)}} \frac{y_1 - y_0}{x_1 - x_0}}$$

(Next Time: examples!)