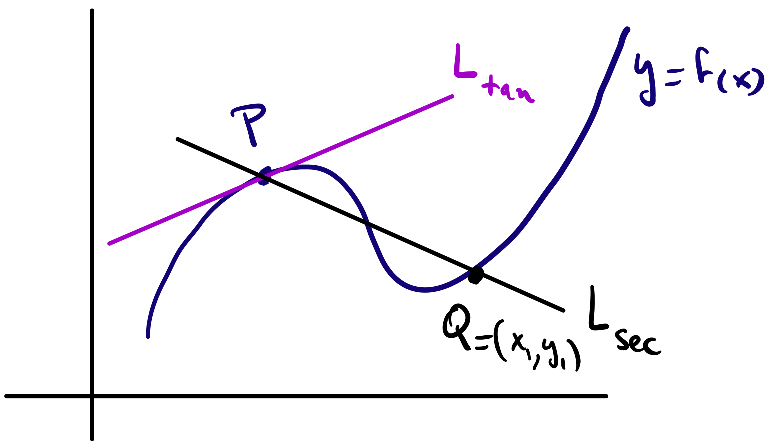
Lecture II: § 2.2 (cont) Slopes of tangents
§ 2.3: Definition of derivatives

Recall: "Tangent lines are limits of secant lines."



- $P=(x_0,y_0)$
- $Q=(x_1,y_1)$ approaching P with $x_1 \neq x_0$

L_{Tan} $\quad Y = m_{Tan}(X-x_0)+y_0$

L_{sec}: $\quad Y = \underbrace{\frac{y_1-y_0}{x_1-x_0}}_{=m_{sec}}(X-x_0)+y_0$

Then $\boxed{m_{tan} := \lim_{(x_1,y_1)\to(x_0,y_0)} \frac{y_1-y_0}{x_1-x_0}}$ slope of the tangent line at P (provided it exists!)

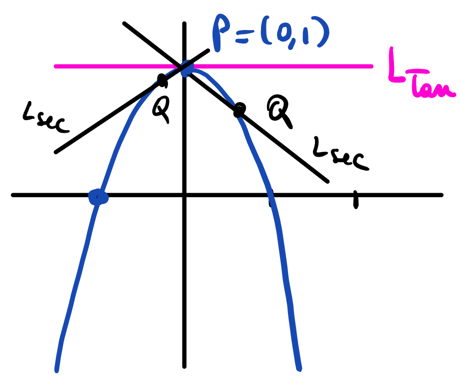
Important fact: We ALWAYS take $Q \neq P$

If the curve is the graph of a function, so $y=f(x)$ we necessarilly have $x_1 \neq x_0$ (otherwise, $y_1 = y_0 = f(x_0)$)

We use m_{Tan} to definite the derivative of f at x_0.

Def: $m_{tan} = \lim_{\substack{x_1\to x_0 \\ (f(x_1)\to f(x_0))}} \frac{f(x_1)-f(x_0)}{x_1-x_0} =: \boxed{f'(x_0)}$ (usually $f(x_1)\to f(x_0)$ will be automatically true for nice functions)

Numerical examples: $\quad y = 1-x^2 =: f(x)$



Ex 1: $P=(0,1)$ $\rightsquigarrow$ L_{Tan}: $Y = 0(x-0)+1$ $\boxed{Y=1}$

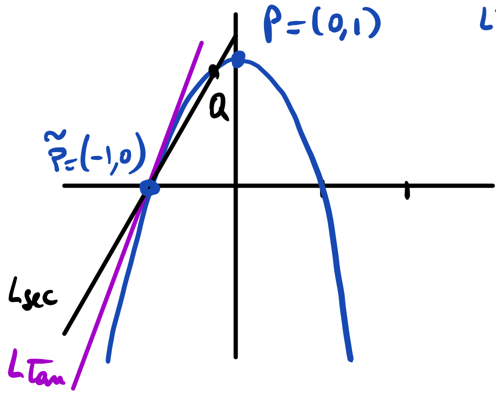
$$m_{Tan} = \lim_{x_1\to 0} \frac{(1-x_1^2)-1}{x_1-0} = \lim_{x_1\to 0} \frac{-x_1^2}{x_1} \underset{x_1\neq 0}{=} \lim_{x_1\to 0} -x_1 = 0$$

Ex 2: $P=(-1,0) \leadsto L_{Tan}$: $Y = 2(X-(-1))$

$\boxed{Y = 2X+2}$

$$m_{tan} = \lim_{x_1\to -1} \frac{(1-x_1^2)-(1-(-1)^2)}{x_1-(-1)} = \lim_{x_1\to -1} \frac{1-x_1^2}{x_1+1}$$

$$= \lim_{x_1\to -1} \frac{(1-x_1)(1+x_1)}{x_1+1} \underset{x_1=-1}{=} \lim_{x_1\to -1} 1-x_1 = 2$$



Conclusions: • Geometry tells us what the Tangent line should be. (guess)

• "Calculus" (or Analysis) allows us to formally compute it (certificate)

(How? (1) Choose the pt $P=(x_0,y_0)$

(2) Compute the slope m_{tan} as a limit of m_{sec}

(3) Write down the equation for L_{tan}: $y = m_{tan}(x-x_0)+y_0$.

Formal procedure: "Method of increments"

- Think of $P=(x_0,y_0)$ as being fixed
- Move the point $Q=(x_1,y_1)$ towards P.

Write $\boxed{\begin{aligned} x_1 &= x_0+\Delta x \\ y_1 &= y_0+\Delta y \end{aligned}}$ where Δx & Δy are small increments (positive or negative!) that we let go to 0.

$$\leadsto m_{sec} = \frac{y_1-y_0}{x_1-x_0} = \frac{\Delta y}{\Delta x} \quad \text{so} \quad m_{tan} = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x}$$

. Special case: $y=f(x)$ so $y_0 = f(x_0)$ & $y_1 = f(x_1)$

Substitution gives $\Delta y = y_1 - y_0 = f(x_1) - f(x_0) = f(x_0+\Delta x) - f(x_0)$

Conclusion:
$$\boxed{\begin{aligned} m_{sec} &= \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \\ m_{tan} &= \lim_{\Delta x\to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \end{aligned}}$$

Earlier examples: $y = 1-x^2$ $\quad P=(0,1)$ $\quad \tilde{P}=(-1,0)$

$$m_{sec} = \frac{1-(x_0+\Delta x)^2-(1-x_0^2)}{\Delta x} = \frac{\cancel{1}-(\cancel{x_0^2}+(\Delta x)^2+2x_0\Delta x)-\cancel{1}+\cancel{x_0^2}}{\Delta x}$$

$$= \frac{-(\Delta x)^2-2x_0\Delta x}{\Delta x} = \frac{-\Delta x(\Delta x+2x_0)}{\Delta x} \underset{\Delta x\neq 0}{=} -(\Delta x+2x_0)$$

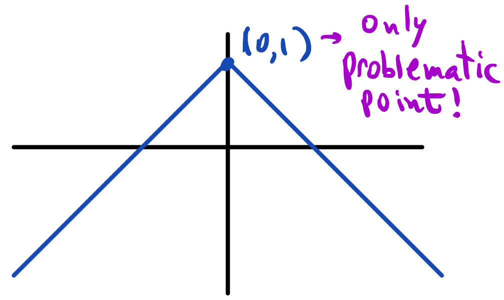
EX 1: $P=(0,1)$ $\quad m_{sec} = -(\Delta x+2\cdot 0) = -\Delta x \xrightarrow[\Delta x\to 0]{} 0 = m_{Tan}$ ✓

(same answers as before!)

Ex 2 $\tilde{P}=(-1,0)$ $\quad m_{sec} = -(\Delta x+2\cdot(-1)) = -\Delta x+2 \xrightarrow[\Delta x\to 0]{} 2 = m_{Tan}$ ✓

An example with no tangent:

$$y=f(x) = 1-|x| = \begin{cases} 1-x & x>0 \\ 1+x & x<0 \end{cases}$$



- All points except $(0,1)$ admit a tangent
- Increment method gives $L_{tan}: y=x+1$ for any $P=(x_0,y_0)$ with $x_0<0$,

  $L_{Tan}: y=-x+1$ ———————— $x_0>0$.

- At $P=(0,1)$ $\quad m_{Tan} = \lim_{\Delta x\to 0} \frac{(1-|0+\Delta x|)-1}{\Delta x} = \lim_{\Delta x\to 0} \frac{-|\Delta x|}{\Delta x}$

But $\frac{-|\Delta x|}{\Delta x} = \begin{cases} -1 & \Delta x>0 \\ 1 & \Delta x<0 \end{cases}$ so the limit cannot exist. (side limits are different)

§ 2.3 Derivative of a function:

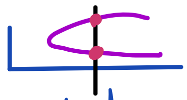
Q1: What is a function?

Definition: A function $g: D\longrightarrow \mathbb{R}$ defined on a set of real numbers D is a formula / rule / law of correspondence that assigns a single real number y to each number x in D. Write $\boxed{y=g(x) \text{ or } x\longmapsto g(x)}$

Call D = domain of g, $\quad x$ = independent variable
$\quad y$ = dependent variable

Name: $Im(g)$ = Range of g = {values y that get assigned by g}

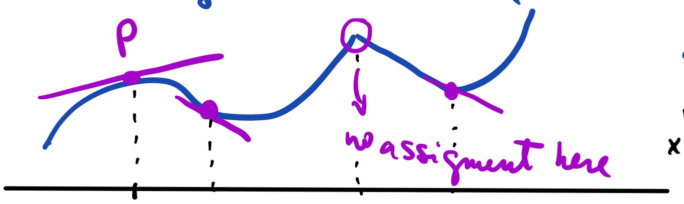
Recall We certify if a curve is the graph of a function via the vertical line test:



2 pts in the intersection, so the horiz parabola fails the test.

Example 1: $g(x) = 1-|x|$ $Im(g) = \{y \text{ in } \mathbb{R} \text{ with } y \leq 1\}$

Example 2 Fix a curve $\mathcal{C}$ & assign to each P the slope of the tangent line to $\mathcal{C}$ at P, whenever possible.



$P \longmapsto m_{Tan}$ ("slope function")

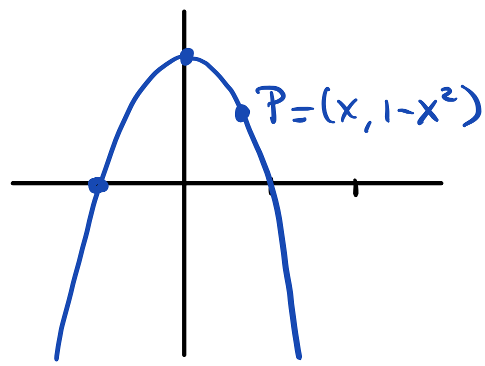
$\|$
x-coord of P

- If $\mathcal{C}$ is the graph of a function $y=f(x)$, then the slope fnc is $x_0 \longmapsto \lim\limits_{\Delta x \to 0} \dfrac{f(x_0+\Delta x) - f(x_0)}{\Delta x} =: f'(x_0)$

We call it the derivative of f.

Note: Since tangent lines not always exist, the domain of f' might be smaller than the domain of f

Example 1: $f(x) = 1-x^2$ Domain $= \mathbb{R}$



$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$$

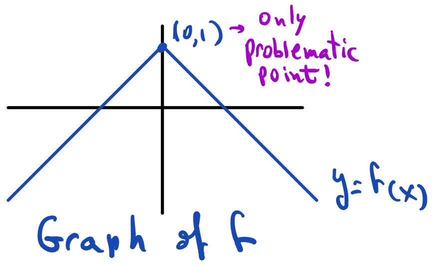
$$= \lim_{\Delta x \to 0} \frac{1-(x_0+\Delta x)^2 - (1-x_0^2)}{\Delta x}$$

done on page 3

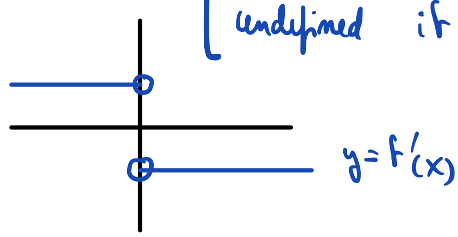
$$= \lim_{\Delta x \to 0} -(\Delta x + 2x_0) = -2x_0.$$

$\rightsquigarrow$ f' is defined for any x_0 & its formula is $f'(x) = -2x$

Example 2: $f(x) = 1-|x|$ $\rightsquigarrow$ $f'(x) = \begin{cases} 1 & \text{if } x<0 \\ -1 & \text{if } x>0 \\ \text{undefined} & \text{if } x=0 \end{cases}$ $\rightsquigarrow$ Domain $= \{x \neq 0\}$



$\rightsquigarrow$



Graph of f'

Notation: Leibniz

- $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \frac{df}{dx} = \frac{d}{dx} f$$

Think: "Operation $\frac{d}{dx}$ is performed on f".

We see how correct notation can clarify concepts!

Exercise: Compute $f'(x)$ for $f(x) = x^3$ (any x), $f(x) = \frac{1}{x}$ ($x \neq 0$) & $f(x) = \sqrt{x}$ ($x \geq 0$)