Lecture III: §2.3 The concept of a limit

Recall: Given a function $f : D \longrightarrow \mathbb{R}$ ← set of all real numbers, where D is a subset of $\mathbb{R}$ (write $D \subseteq \mathbb{R}$), we define its derivative f' as a new function with formula

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

This is defined whenever the limit exists and x is in D.

Examples ① If $f(x) = 1 - x^2$, then $f'(x) = -2x$. (last Time)

So the domain of both f & f' is all $\mathbb{R}$.

② $f(x) = \frac{1}{x}$ for $x \neq 0$

$$\leadsto f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x}$$

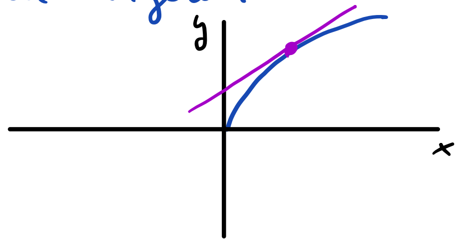
$$= \lim_{\Delta x \to 0} \frac{\frac{x - (x+\Delta x)}{(x+\Delta x)\, x}}{\Delta x} = \lim_{\Delta x \to 0} \frac{-\Delta x}{\Delta x\,(x+\Delta x)\, x}$$

$$\underset{\Delta x \neq 0}{=} \lim_{\Delta x \to 0} \frac{-1}{(x+\Delta x)\, x} = \frac{-1}{x^2}$$

. This is defined for $x \neq 0$.

So Domain of f = Domain of f' = all nonzero real numbers.

③ $f(x) = \sqrt{x}$ for $D = \{x \geq 0\}$



Claim $f'(x) = \frac{1}{2\sqrt{x}}$ defined for $x > 0$

($\leadsto$ Domain of $f \neq$ Domain of f')

Why? $f'(x) = \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x}$

Trick: Multiply by $1 = \frac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}}$, so

$$f'(x) = \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}} = \lim_{\Delta x \to 0} \frac{(x+\Delta x) - x}{\Delta x\left(\sqrt{x+\Delta x} + \sqrt{x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x\left(\sqrt{x+\Delta x} + \sqrt{x}\right)} \underset{\Delta x \neq 0}{=} \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}} \underset{x \neq 0}{=} \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Notation: Leibniz

- : $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$

$f'_{(x)} = \lim\limits_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim\limits_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \frac{df}{dx} = \left(\frac{d}{dx}\right) f$

Think: "Operation $\frac{d}{dx}$ is performed on f".

We see how correct notation can clarify concepts!

- So far, we have USED limits, but never really defined them precisely...

(We've only relied on our intuition of what a limit should be...)

§1 Limits:

Q: What does $\lim\limits_{x \to a} g(x) \overset{x = a + \Delta x}{=} \lim\limits_{\Delta x \to 0} g(a+\Delta x) = L$ mean?

We need: (1) A function $g: D \to \mathbb{R}$ defined around a (not necessarily at a!)

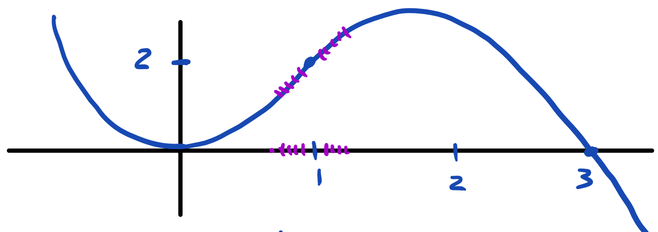
(2) A number L

Ideas for defining the limit:

(*) As x approaches a, the value of $g(x)$ approaches L

(**) We can make $g(x)$ be as close as we want to L by taking x closed enough to a. "not very precise!"

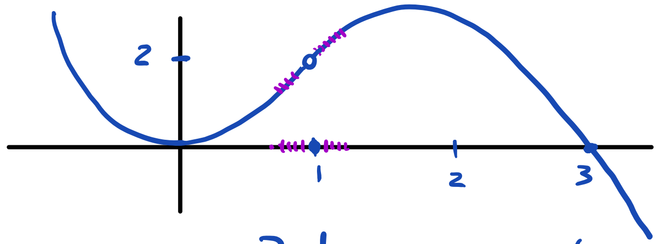
Important remark: We are not claiming anything about the value $g(a)$ (g may very well not be defined at a), but we shouldn't care about $g(a)$ after all!



Examples ① $g(x) = 3x^2 - x^3$

Can guess $\lim\limits_{x \to 1} g(x) = \lim\limits_{\Delta x \to 0} g(1+\Delta x) = g(1) = 2$ from the graph. Same for other values of a ⇝ $\lim_{x\to a} g(x) \underset{\substack{\downarrow\\ x=a+\Delta x}}{=} \lim_{\Delta x\to 0} g(a+\Delta x) = g(a)$

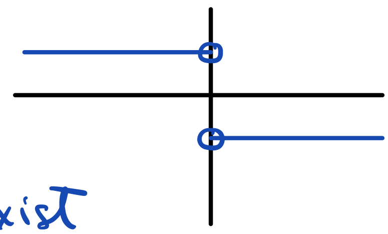
(2) $g(x) = \begin{cases} 3x^2 - x^3 & \text{if } x\neq 1 \\ 0 & \text{if } x=1 \end{cases}$



Still $\lim_{x\to 1} g(x) = \lim_{\Delta x\to 0} g(1+\Delta x) = 2$ But now $2 \neq g(1)$

[Away from $x=1$ the graph of the functions from (1) & (2) look the same, so the limits must be the same!]

(3) $f(x) = \dfrac{-|x|}{x}$ for $x\neq 0$ ⇝ $\begin{cases} -1 & \text{if } x>0 \\ 1 & \text{if } x<0 \end{cases}$



$\lim_{x\to 1} f(x) = -1$, $\lim_{x\to 0} f(x)$ does not exist
($f(x)$ approaches 1 from the left & -1 from the right!)

Q: What can we say if we can't draw the graph of g?

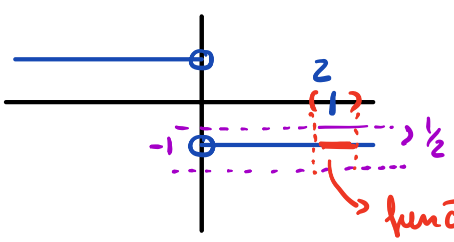
A: We make (**) more precise ⇝ We characterize limits as a GAME of CHOICE

(1) YOU pick how close to L you want to be (say, for example, $|g(x) - L| < 10^{-8}$)

(2) YOU challenge ME to determine how close to a, I should pick x to be (say $0<|x-a| < 10^{-26}$) to ENSURE that
$|g(x)-L| < 10^{-8}$ whenever $0<|x-a|<10^{-26}$.

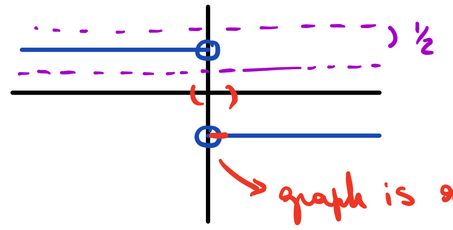
↳ This will change if you pick another error other than 10^{-8}.

Ex (2):

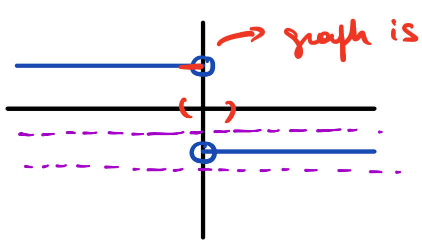


$\lim_{x\to 2} f(x) = -1$

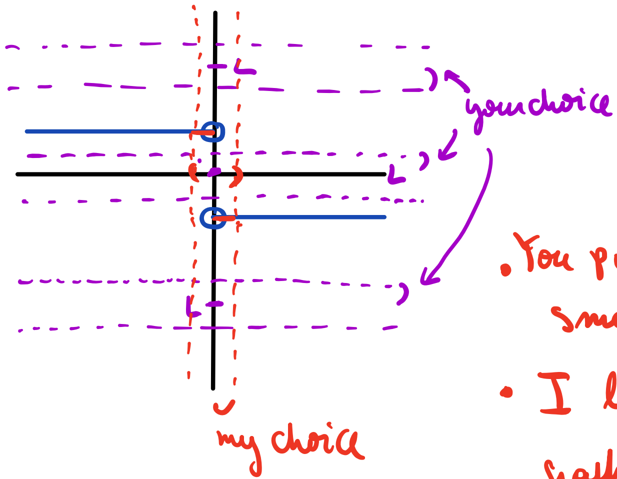
function stays within the horizontal strip around −1 you picked



$\lim_{x\to 0} f(x) \neq 1$ You pick $\frac{1}{2}$ & I loose!



$\lim_{x\to 0} f(x) \neq -1$ Again you pick $\frac{1}{2}$ & I loose!



$\lim_{x\to 0} f(x) \neq L$ for any $L \neq 1, -1$.

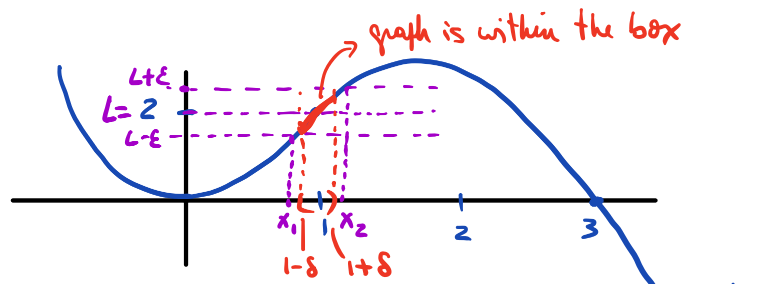
- You pick the width of the horizontal strip around L small enough so that the strip avoids the graph
- I lose! No matter the error I choose, the graph near $a=0$ is outside the strip.

<u>Definition</u>: We say $\lim_{x\to a} g(x) = L$ if for <u>every</u> $\varepsilon>0$ (your choice) there is a $\delta>0$ (my choice) such that: if $0<|x-a|<\delta$ then $|g(x)-L|<\varepsilon$ (In short, I ALWAYS win the game)

<u>Ex</u> 1 $\lim_{x\to 1} 3x^2-x^3 = 2$

Pick $x_1<1$ with $3x_1^2-x_1^3 = L-\varepsilon$

$x_2>1$ — $3x_2^2-x_2^3 = L+\varepsilon$



and $\delta \leq \min\{1-x_1, x_2-1\}$ works

<u>Typically</u>: δ has a formula involving ε & a.

⚠ Even if we have a formula for g, finding δ given ε & a can be very challenging (because it involves "inverting" a formula)

Problem: $g(x) = 1-x^2$ Show $\lim_{x\to 0} g(x) = 1 \,(=L)$ via ε-δ method

Soln: Given any $\varepsilon > 0$, we want to find $\delta > 0$ (in terms of ε) so that if $0 < |x-0| < \delta$, then $|g(x)-1| = |1-x^2-1| = |-x^2| = x^2 < \varepsilon$

Need to pick δ so that $0<|x|<\delta$ forces $x^2 < \varepsilon$

Answer: $\delta = \sqrt{\varepsilon}$ will do (If $0<|x|<\delta$, we square it to get $0 < \underbrace{|x|^2}_{=x^2} < \delta^2 = \varepsilon$ ✓)