Lecture IV §2.5 ε/δ-definition of limits
2 Trigonometric limits
§2.4 Rates of change & velocity.

Recall: We saw $\lim_{x\to a} g(x) = L$ as a game of choice where I win.

(1) You pick how close to L the value of $g(x)$ has to be (say $|g(x)-L|<10^{-8}$)

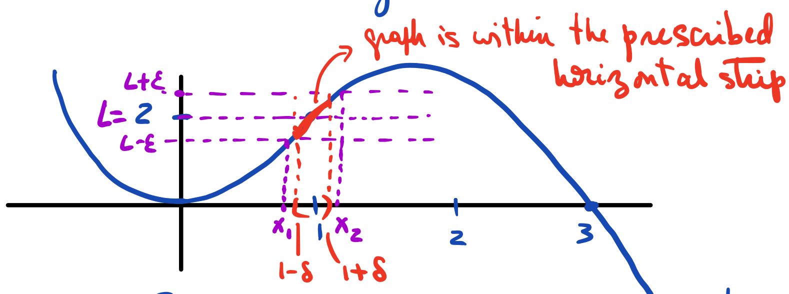
(2) I pick a threshold within a for the variable x to be (say 10^{-26}) so that if $0<|x-a|<10^{-26}$ (that is, $a-10^{-26}<x<a+10^{-26}$ & $x\neq a$) then we MUST have $|g(x)-L|<10^{-8}$ (That is, $L-10^{-8}<g(x)<L+10^{-8}$)

(If I can always pick, then I win. Otherwise you do & $\lim_{x\to a} g(x)\neq L$)

§1 Formal definition of limit:

Definition: We say $\lim_{x\to a} g(x) = L$ if for every $\varepsilon>0$ (your choice) there is a $\delta>0$ (my choice) such that: if $0<|x-a|<\delta$ then $|g(x)-L|<\varepsilon$ (In short, I ALWAYS win the game)

Ex ① $\lim_{x\to 1} 3x^2-x^3 = 2$



Pick $x_1<1$ with $3x_1^2-x_1^3 = L-\varepsilon$

$x_2>1$ — $3x_2^2-x_2^3 = L+\varepsilon$ $\implies$ $\delta \leq \min\{1-x_1, x_2-1\}$ works

Typically: δ has a formula involving ε & a.

⚠ Even if we have a formula for g, finding δ given ε & a can be very challenging (because it involves "inverting" a formula)

Problem: $g(x) = 1-x^2$ Show $\lim_{x\to 0} g(x) = 1\ (=L)$ via ε-δ method

Soln: Given any $\varepsilon>0$, we want to find $\delta>0$ (in terms of ε) so that if $0<|x-0|<\delta$, then $|g(x)-1| = |1-x^2-1| = |-x^2| = x^2<\varepsilon$

Need to pick δ so that $0<|x|<\delta$ forces $x^2<\varepsilon$

Answer: $\delta=\sqrt{\varepsilon}$ will do (If $0<|x|<\delta$, we square it to get $0<\underbrace{|x|^2}_{=x^2}<\delta^2=\varepsilon$ ✓)

Properties (Limit Laws) Assume that f & g are two functions defined around a with $\lim_{x\to a} f(x) = L$ & $\lim_{x\to a} g(x) = M$. Then:

(1) $\lim_{x\to a} (f(x) \pm g(x)) = L \pm M$ ($f \pm g$ are functions defined near a)

(2) $\lim_{x\to a} f(x)\, g(x) = LM$ ($f \cdot g$ is a new function defined near a)

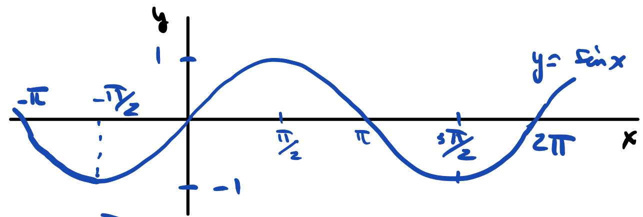
(3) For any real number c, we have $\lim_{x\to a} c f(x) = cL$ (cf is a new function defined near a)

(4) If $M \neq 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$ ($\frac{f}{g}$ is a new function defined near a)

Next Time: we'll discuss why these properties hold (Appendix A2)

§2 Two trigonometric limits:

EXAMPLE 1: $\lim_{h\to 0} \frac{\sin(h)}{h} = ?$

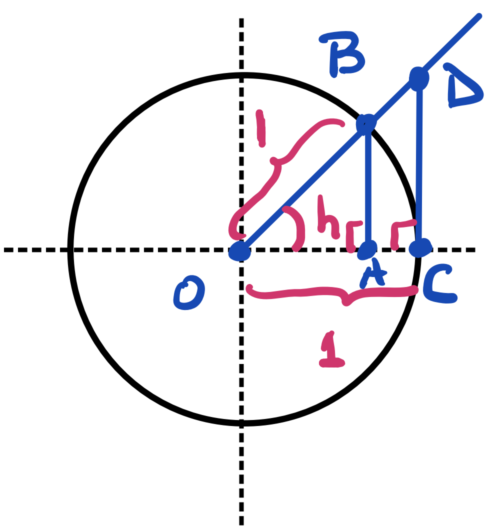


Note $\sin(0) = 0$ so we have a $\frac{0}{0}$-indeterminacy.

Q: Do things cancel out like with $\frac{x^2}{x}$, $\frac{x}{x^2}$ or $\frac{x}{x}$?

A To answer, we use trigonometry in the unit circle.

Fix $h > 0$ (otherwise use $\frac{\sin(-h)}{-h} = \frac{-\sin h}{-h} = \frac{\sin h}{h}$)



- We draw the sector of the circle with $\measuredangle = h$. This determines the point B
- Next we draw 2 right triangles $\overset{\triangle}{OBA}$ & $\overset{\triangle}{ODC}$. They are similar (all 3 angles match pairwise: $\measuredangle BOA = \measuredangle DOC$
  $\measuredangle OAB = \measuredangle OCD = 90^\circ$
  $\measuredangle OBA = \measuredangle ODC$)

. The ratios of the sides are determined by various trigonometric functions

$$\frac{AB}{1} = \sin(h) = \frac{CD}{OD} \quad \rightsquigarrow AB = \sin h$$

$$\frac{CD}{1} = \frac{BA}{OA} = \tan(h) \quad \rightsquigarrow CD = \tan h$$

$$\frac{OA}{1} = \cos(h) = \frac{1}{OD} \quad \rightsquigarrow OA = \cos h$$

$\rightsquigarrow$ Ingredients: . arc of the circle $= \overset{\frown}{BC}$

- sector OBC of the circle
- 2 triangles $\overset{\triangle}{OAB}$ & $\overset{\triangle}{OCD}$.

$$\text{Area of } \overset{\triangle}{OBC} = \text{area of circle} \cdot (\text{fraction of angle with respect to } 2\pi)$$

$$= \pi \cdot 1^2 \cdot \frac{h}{2\pi} = \frac{h}{2} \qquad \left(= \frac{|h|}{2} \text{ if } h<0\right)$$

$$\text{Area of } \overset{\triangle}{OAB} = \frac{1}{2}\,\text{base} \cdot \text{height} = \frac{1}{2}\, OA \cdot AB$$

$$= \frac{1}{2}\cos(h) \cdot \sin(h) \qquad \left(= \frac{\cos(h)\,\sin(|h|)}{2} \text{ if } h<0\right)$$

$$\text{Area of } \overset{\triangle}{OCD} = \frac{1}{2}\,\text{base} \cdot \text{height} = \frac{1}{2}\, OC \cdot CD$$

$$= \frac{1}{2}\tan(h) = \frac{\sin(h)}{2\cos(h)} \qquad \left(= \frac{\sin(|h|)}{2\cos(h)} \text{ if } h<0\right)$$

So $\quad \text{Area of } \overset{\triangle}{OAB} \leq \text{Area of } \overset{\triangle}{OBC} \leq \text{Area of } \overset{\triangle}{OCD}$

$$\frac{\cos(h)\sin(h)}{2} \leq \frac{h}{2} \leq \frac{\sin(h)}{2\cos(h)}$$

$$\left(\frac{\cos h \ \sin(|h|)}{2} \leq \frac{|h|}{2} \leq \frac{\sin(|h|)}{2\cos(h)} \quad \text{if } h<0 \right.$$

$$-\frac{\cos(h)\sin(h)}{2} \leq \frac{-h}{2} \leq \frac{-\sin h}{2\cos h}$$

. If $h>0$, we divide all 3 expressions by $\frac{h}{2}>0$ & the inequalities remain: We get

$$\cos h \ \frac{\sin h}{h} \leq 1 \leq \frac{\sin h}{h}\ \frac{1}{\cos h}$$

So $\boxed{\cos h \leq \frac{\sin h}{h} \leq \frac{1}{\cos h}}$ (*) $\quad$ $\cos(0)=1$ & $\cos h > 0$ for h near 0)

- If $h<0$, we divide by $\frac{-h}{2} > 0$ & the inequalities remain:

We get: $\cos h \frac{\sin h}{h} \leq 1 \leq \frac{\sin h}{h} \frac{1}{\cos h}$ so once again we get (*)

<u>Conclusion</u>: $\cos h \leq \frac{\sin h}{h} \leq \frac{1}{\cos h}$ & we know that $\lim\limits_{h\to 0} \cos h = 1 = \lim\limits_{h\to 0} \frac{1}{\cos h}$. So $\frac{\sin h}{h}$ is <u>squeezed between</u> two functions with the same limit. This forces:

$$\boxed{\lim_{h\to 0} \frac{\sin h}{h} = 1}$$

<u>Application</u>: $\lim\limits_{h\to 0} \frac{\cos h - 1}{h} = ?$ (also $0/0$-indeterminacy)

We use a well-known Trig identity: $\cos^2 h + \sin^2 h = 1$

$$\frac{\cos h - 1}{h} = \frac{(\cos h - 1)}{h} \frac{(\cos h + 1)}{\cos h + 1} = \frac{\cos^2 h - 1}{h(\cos h + 1)} \overset{\downarrow}{=} \frac{-\sin^2 h}{h(\cos h + 1)}$$

$$= \frac{\sin h}{h} \left(\frac{-\sin h}{\cos h + 1}\right)$$

So $\lim\limits_{h\to 0} \frac{\cos h - 1}{h} = \lim\limits_{h\to 0} \underbrace{\frac{\sin h}{h}}\ \underbrace{\left(\frac{-\sin h}{\cos h + 1}\right)} \underset{\text{Limit Laws}}{=} 1 \cdot 0 = \boxed{0}$

2 functions with known limits.

<u>Know</u> $\lim\limits_{h\to 0} \frac{\sin h}{h} = 1$

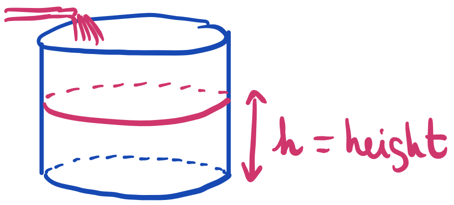
$$\lim_{h\to 0} \frac{(-\sin h)}{\cos h + 1} = \frac{-\sin(0)}{(\cos 0) + 1} = \frac{-0}{1+1} = \frac{-0}{2} = 0$$

<u>§ 2.4 Velocity & rates of change</u>:

Think of $y = f(x)$ as indicating a relationship between 2 physical quantities, so $f' = dy/dx$ is the instantaneous rate of change.

Application 1: The independent variable $x = t$ is TIME.

Example 1: Filling a water tank:



$V(t)$ = Volume of water at Time t

$\frac{dV}{dt}$ = rate of change at which the tank is being filled.

Also $h = h(t)$ (the height of water is changing with time)

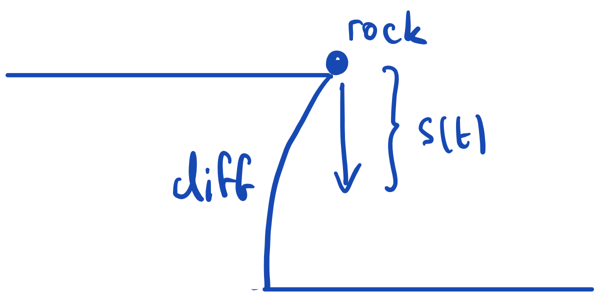
$\leadsto \frac{dh}{dt}$ = rate at which the height changes.

Note: V & h are related $V(t) = h(t)$ Area of the base

this is fixed

so their rates of change are related

Example 2: Rock falling of a cliff:



$s(t)$ = position at Time t

$\frac{ds}{dt} = s'(t) :=$ velocity $= v(t)$

It has a direction (positive or neg.)!

$\frac{dv}{dt} = v'(t) :=$ acceleration $= a(t)$

$v(t)$ = rate of change of position

$a(t)$ = ______ velocity

Speed: $= |v(t)|$ has no direction (value in the dashboard of your car)

Here: experimental results propose $s(t) = 16t^2$ (ft)

$s'(t) = 32t$

ACCELERATION due to gravity $\leadsto$ $a(t) = 32 \ \mathrm{ft}/s^2$ $\left(9.8 \frac{m}{s^2}\right)$

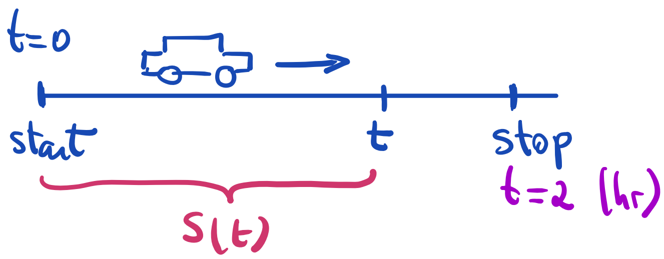
Application 2: The independent variable can be something else,

Eg: x = # cars produced

$C(x)$ = cost of producing x cars.

$\frac{dC}{dx}$ = marginal cost or "cost per unit"

Application 3: Car driving straight down the road.



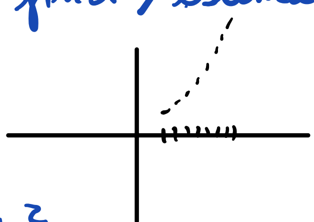
Odometer = measures distance travelled

Speedometer = —— speed (forward velocity)

- Watching odometer gives $s(t)$ = distance at time t
- —— speedometer — $v(t) = s'(t)$ = forward velocity at time t

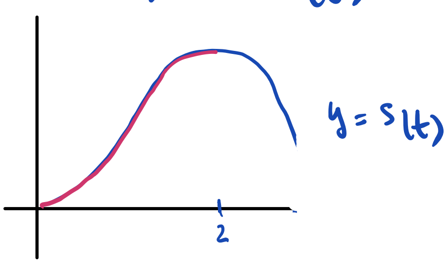
By empirical measures, we find / estimate that

$$s(t) = 60t^2 - 20t^3$$

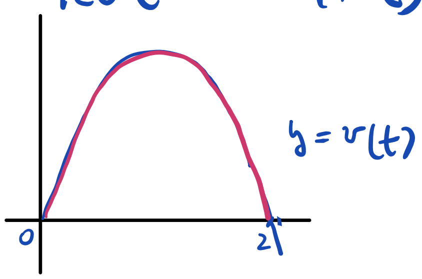


So $v(t) = s'(t) = 120t - 60t^2$ ⇝ we can confirm this with the speedometer

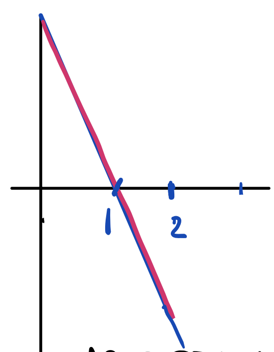
$$a(t) = v'(t) = 120 - 120t = 120(1-t)$$



POSITION (ODOMETER)



FORWARD VELOCITY (SPEEDOMETER)



ACCELERATION

Each graph gives a "different" picture of our travels. (between $t=0$ & $t=2$)