Lecture V Appendix A2 - Theorems about limits

Recall: Given a function $f: D \longrightarrow \mathbb{R}$ defined around $x=a$ (but not necessarily at a) we say $\boxed{\lim_{x\to a} f(x) = L}$ if for EVERY $\varepsilon>0$ (YOUR choice!), we can ALWAYS find a $\delta>0$ (MY choice, usually depends on ε & a) such that if $a-\delta < x < a+\delta$ & $x \neq a$ (in short, $\boxed{0<|x-a|<\delta}$), then we automatically have $L-\varepsilon < f(x) < L+\varepsilon$ (in short, $\boxed{|f(x)-L|<\varepsilon}$)

→ The limit is L if I ALWAYS win the game, independent of your choice

Example: $f(x) = 5x+4$ & $a=0$ **Guess**: $\lim_{x\to 0} 5x+4 = 4$

WANT $|(5x+4)-4|<\varepsilon$, that is $|5x|<\varepsilon$ if $|x|<\delta$

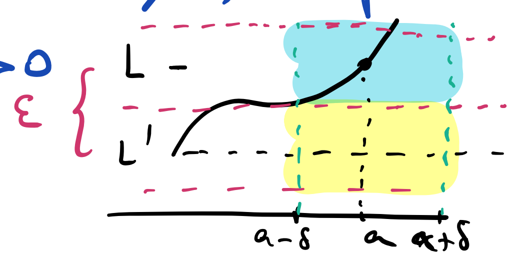
Take $\delta = \frac{\varepsilon}{5}$ or smaller! (reverse engineer!)

Indeed, if $0<|x|<\delta$, then $|5x| < 5\delta = \varepsilon$, so $|(5x+4)-4|<\varepsilon$.

- **Natural Question 1**: Can we have 2 different limits L & L'?
- **Natural Question 2**: How do limits behave with respect to the four standard operations $+, -, \cdot, /$ for real numbers? What about inequalities?

Theorem 1: If $\lim_{x\to a} f(x) = L$ & $\lim_{x\to a} f(x) = L'$, then $L = L'$

Why? We argue by contradiction. We assume $L \neq L'$, so up to symmetry, we suppose $L<L'$. Pick $\varepsilon = L'-L>0$



- By definition, we can find $\delta>0$ so that if $0<|x-a|<\delta$, then $|f(x)-L|<\frac{\varepsilon}{2}$
- ______, ______ $\delta'>0$ ______ $0<|x-a|<\delta'$, ______ $|f(x)-L'|<\frac{\varepsilon}{2}$

Pick $\delta''=\min\{\delta,\delta'\}>0$ & assume $0<|x-a|<\delta''$.

Then $\varepsilon = L'-L = |L'-L| = |L'-f(x)+f(x)-L| \underset{\Delta \text{ ineq.}}{\leq} \underbrace{|L'-f(x)|}_{<\frac{\varepsilon}{2}} + \underbrace{|f(x)-L|}_{<\frac{\varepsilon}{2}}$

We conclude that $\varepsilon = |L'-L| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$ so $\varepsilon<\varepsilon$.

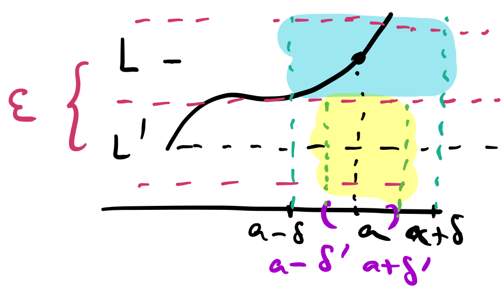
This is a contradiction!

Q: What went wrong?

A: Our original assumption $L\neq L'$ leads to a contradiction ($\varepsilon<\varepsilon$) so it must be FALSE. We conclude from here that $\boxed{L=L'}$.

Remark:
- This is an example of a proof by contradiction (more on MATH 3345)
- We used $|c+d|\leq|c|+|d|$ (Δ-inequality)
- To combine several inequalities, it's useful to take $\min\{\delta_1,\delta_2,\dots\}$

Pictorial proof:



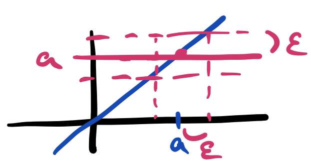
we cannot be in 2 boxes at the same time, since they don't have common points!

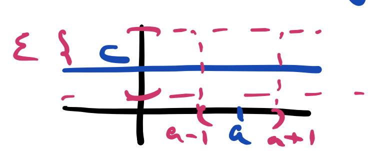
Warm-up limits:

Theorem 2 $\lim_{x\to a} x = a$ & $\lim_{x\to a} c = c$ for any constant c.

Why? Want $|x-a|<\varepsilon$ if $|x-a|<\delta$. Pick $\delta=\varepsilon$

Want $|c-c|=0<\varepsilon$ if $|x-a|<\delta$. Pick any δ, for example $\delta=1$.



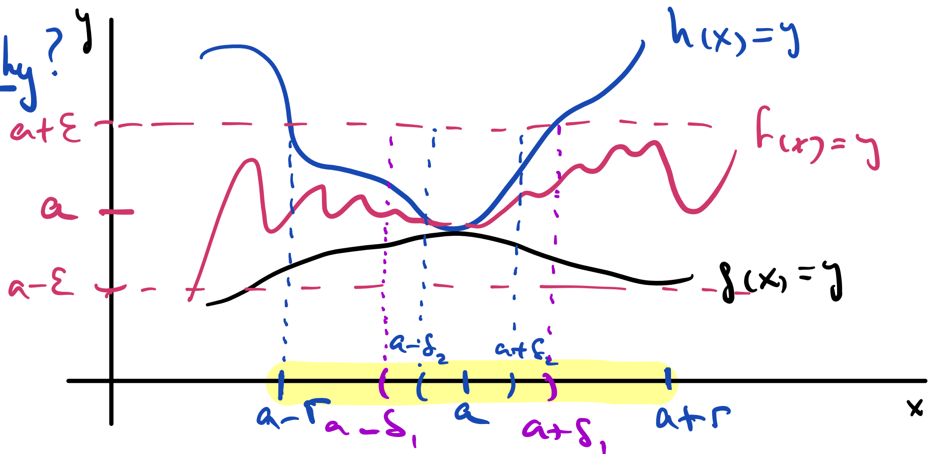


Squeeze Theorem Assume we have 3 functions defined around a, and satisfying $g(x) \le f(x) \le h(x)$ around $x=a$.
If $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$, then the limit $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} f(x) = L$

Recall: Last Time we used $\boxed{\cos h \le \frac{\sin h}{h} \le \frac{1}{\cos h}}$ for $h>0$

to show $\lim_{h\to 0} \frac{\sin h}{h} = 1$.

Why?



We use ε/δ for $f(x)$

- Assume $g(x) \le f(x) \le h(x)$ if $0<|x-a|<r$ for some $r>0$

Fix $\varepsilon > 0$

- Pick $\delta_1 > 0$ so that if $0<|x-a|<\delta_1$, then $|h(x) - L| < \varepsilon$
- —— $\delta_2 > 0$ ———— $0<|x-a|<\delta_2$ —— $|g(x) - L| < \varepsilon$

Pick $\delta = \min\{\delta_1, \delta_2, r\} > 0$. & assume $0<|x-a|<\delta$

Then

$$\underset{\delta\le\delta_1}{L-\varepsilon <} \; g(x) \underset{\delta\le r}{\le} f(x) \underset{\delta\le r}{\le} h(x) \underset{\delta<\delta_2}{<} L+\varepsilon$$

So by looking at the ends, we get $L-\varepsilon \le f(x) \le L+\varepsilon$,
that is $|f(x) - L| < \varepsilon$.

Conclusion: we found $\delta > 0$ satisfying:
if $0<|x-a|<\delta$, then automatically $|f(x)-L|<\varepsilon$.

It follows that $\lim_{x\to a} f(x) = L$, as we wanted to show. $\square$