Lecture VI Appendix A2 - More Theorems about limits

§2.6 Continuous functions

§1. Limits Laws:

Theorem 1 (Limits and arithmetic operations). Fix two functions $f(x), g(x)$ defined around $x=a$ & assume that $\lim_{x\to a} f(x) = L$ & $\lim_{x\to a} g(x) = M$.

Then (1) $\lim_{x\to a} (f(x)+g(x)) = L+M$

(2) $\lim_{x\to a} (f(x)-g(x)) = L-M$

(3) $\lim_{x\to a} f(x)\,g(x) = L\cdot M$

Why? As usual, we use ε/δ-method

(1) & (2): Pick any $\varepsilon>0$ & take $\varepsilon' = \frac{\varepsilon}{2}$

- Since $\lim_{x\to a} f(x) = L$, we can find $\delta_f>0$ so that if $0<|x-a|<\delta_f$, then we must have $|f(x)-L|<\varepsilon' = \frac{\varepsilon}{2}$

- Since $\lim_{x\to a} g(x) = M$, we can find $\delta_g>0$ so that if $0<|x-a|<\delta_g$, then we must have $|g(x)-M|<\varepsilon' = \frac{\varepsilon}{2}$

Pick $\boxed{\delta = \min\{\delta_f, \delta_g\}>0}$

- Then: $|f(x)+g(x)-(L+M)| \underset{\text{reorder}}{=} |\underbrace{f(x)-L}_{=b} + \underbrace{g(x)-M}_{=c}| \leq$

$|b+c|\leq |b|+|c|$ (see page 7)

$\leq |f(x)-L| + |g(x)-M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$0<|x-a|<\delta<\delta_f$
$0<|x-a|<\delta<\delta_g$

Conclude: If $0<|x-a|<\delta$, then $|(f(x)+g(x))-(L+M)|<\varepsilon$

So $\lim_{x\to a} f(x)+g(x) = L+M$ by ε/δ-definition.

- Similarly: $|f(x) - g(x) - (L - M)| = |\underbrace{(f(x) - L)}_{=b} - \underbrace{(g(x) - M)}_{=c}|$

$$\underset{\substack{\downarrow \\ |b-c| \leq |b|+|c| \\ \text{(see page 7)}}}{\leq} |f(x) - L| + |g(x) + M| \underset{\substack{\downarrow \\ |x-a|<\delta<\delta_f \\ |x-a|<\delta<\delta_g}}{<} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Conclude: If $0<|x-a|<\delta$, then $|(f(x) - g(x)) - (L - M)| < \varepsilon$

So $\lim_{x \to a} f(x) - g(x) = L \quad M$ by ε/δ-definition.

(3) This one is a bit more subtle. Again, pick $\varepsilon > 0$.

$$|f(x)g(x) - LM| \underset{\substack{\downarrow \\ +\,\&\,-\ f(x)M}}{=} |f(x)g(x) - f(x)M + f(x)M - LM|$$

$$\underset{\substack{\downarrow \\ \text{reorder}}}{=} |\underbrace{f(x)(g(x) - M)}_{=b} - \underbrace{M(f(x) - L)}_{=c}|$$

$$\underset{\substack{\downarrow \\ |b-c| \leq |b|+|c|}}{\leq} |f(x)(g(x)-M)| + |M(f(x)-L)|$$

$$\leq |(f(x)-L)(g(x)-M) + L(g(x)-M)| + |M|\,|f(x)-L|$$

$$\leq |f(x)-L|\,|g(x)-M| + |L|\,|g(x)-M| + |M|\,|f(x)-L|$$

$$\underset{\substack{\downarrow \\ |L| \leq |L|+1 \\ |M| \leq |M|+1}}{\leq} |f(x)-L|\,|g(x)-M| + \underbrace{(|L|+1)}_{\leq N}|g(x)-M| + \underbrace{(|M|+1)}_{\leq N}|f(x)-L|$$

Pick $N = \max\{|L|+1, |M|+1\}$, then $N > 0$ &

$$|f(x)g(x) - LM| \leq \underset{(\mathrm{I})}{\boxed{|f(x)-L|\,|g(x)-M|}} + \underset{(\mathrm{II})}{\boxed{N(|g(x)-M| + |f(x)-L|)}}$$

Q: Can we pick $\delta > 0$ so that if $0<|x-a|<\delta$, then $(\mathrm{I}) < \frac{\varepsilon}{2}$ & $(\mathrm{II}) < \frac{\varepsilon}{2}$? If so, we win!

• First, we find δ' working for (I): Take $\varepsilon' = \frac{\sqrt{\varepsilon}}{\sqrt{2}}$

Since $\lim_{x\to a} f(x) = L$ & $\lim_{x\to a} g(x) = M$, we can find $\delta_1', \delta_2' > 0$

so that if $0 < |x-a| < \delta_1'$ then $|f(x) - L| < \varepsilon' = \frac{\sqrt{\varepsilon}}{\sqrt{2}}$

$0 < |x-a| < \delta_2'$ — $|g(x) - M| < \varepsilon' = \frac{\sqrt{\varepsilon}}{\sqrt{2}}$

Pick $\delta' = \min\{\delta_1', \delta_2'\} > 0$. Then if $0 < |x-a| < \delta'$,

we get $(\text{I}) = |f(x) - L|\,|g(x) - M| < \frac{\sqrt{\varepsilon}}{\sqrt{2}} \cdot \frac{\sqrt{\varepsilon}}{\sqrt{2}} = \frac{\varepsilon}{2}$

$\delta' \le \delta_1'$
$\delta' \le \delta_2'$

• Next, we find δ'' working for (II): Take $\varepsilon'' = \frac{\varepsilon}{4N}$

Since $\lim_{x\to a} f(x) = L$ & $\lim_{x\to a} g(x) = M$, we can find $\delta_1'', \delta_2'' > 0$

so that if $0 < |x-a| < \delta_1''$ then $|f(x) - L| < \varepsilon'' = \frac{\varepsilon}{4N}$

$0 < |x-a| < \delta_2''$ — $|g(x) - M| < \varepsilon'' = \frac{\varepsilon}{4N}$

Pick $\boxed{\delta'' = \min\{\delta_1'', \delta_2''\} > 0.}$ Then if $0 < |x-a| < \delta''$, then

we get $(\text{II}) = N\big(|f(x) - L| + |g(x) - M|\big) \le N(\varepsilon'' + \varepsilon'') = N\,2\varepsilon'' = N\,2\frac{\varepsilon}{4N} = \frac{\varepsilon}{2}$.

Pick $\delta = \min\{\delta', \delta''\} > 0$ & conclude that if $0 < |x-a| < \delta$,

then we get $|f(x)g(x) - LM| \le (\text{I}) + (\text{II}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. ☺

<u>Summary</u>: ε/δ-techniques = "The art of bounding"

<u>Important consequence</u>: Any polynomial $f(x) = c_n x^n + c_{n-1}x^{n-1} + \dots + c_1 x + c_0$

with real coefficients $c_n, c_{n-1}, \dots, c_1, c_0$ satisfies

$$\lim_{x\to a} f(x) = c_n a^n + c_{n-1} a^{n-1} + \dots + c_1 a + c_0 = f(a).$$

This is partly why polynomials are our favorite functions!

Theorem 2: If $\lim_{x\to a} g(x) = M \neq 0$, then $\lim_{x\to a} \frac{1}{g(x)} = \frac{1}{M}$.

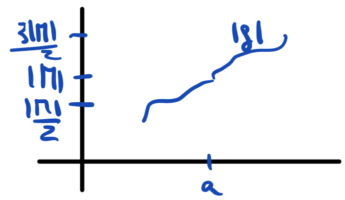
Moreover if $\lim_{x\to a} f(x) = L$ & $M \neq 0$ we get $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$.

Why? • The second claim follows by writing $\frac{f(x)}{g(x)} = f(x)\cdot\frac{1}{g(x)}$ & using the product rule ((3) in Theorem 1).

• To argue for the first statement, we use ε/δ-method.

Write $\left|\frac{1}{g(x)} - \frac{1}{M}\right| = \frac{|M-g(x)|}{|g(x)M|} = |g(x)-M| \ \frac{1}{|M|} \ \frac{1}{|g(x)|}$

Look at the 2 factors separately:



Since $M \neq 0$ we know $|M| > 0$. Pick $\varepsilon' = \frac{|M|}{2} > 0$

• For $\frac{1}{g(x)}$: we can find $\delta_1 > 0$ so that if $0 < |x-a| < \delta_1$, then

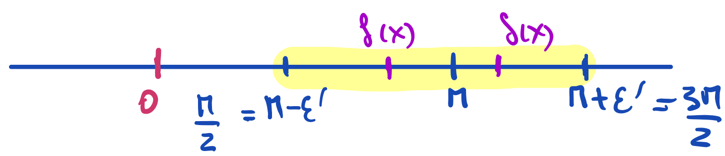
$$|g(x) - M| = \varepsilon' = \frac{|M|}{2} \quad \text{This means}$$

$$M - \frac{|M|}{2} = M - \varepsilon' < g(x) < M + \varepsilon' = M + \frac{|M|}{2}$$

If $M > 0$. $M - \frac{|M|}{2} = M - \frac{M}{2} = \frac{M}{2} = \frac{|M|}{2}$

. $M + \frac{|M|}{2} = M + \frac{M}{2} = \frac{3M}{2} = 3\frac{|M|}{2}$

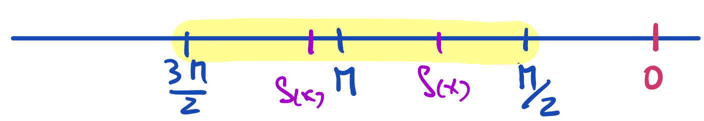
so $0 < \frac{|M|}{2} < g(x) < \frac{3|M|}{2}$ gives $\frac{3|M|}{2} > |g(x)| > \frac{|M|}{2}$



If $M < 0$. $M - \frac{|M|}{2} = M - \frac{(-M)}{2} = \frac{3M}{2}$ & $\frac{M}{2} > \frac{3M}{2}$

. $M + \frac{|M|}{2} = M + \frac{(-M)}{2} = \frac{M}{2}$

so $\frac{3M}{2} < g(x) < \frac{M}{2} < 0$ gives $\frac{3|M|}{2} > |g(x)| > \frac{|M|}{2}$



In both cases: $\dfrac{1}{\frac{3|M|}{2}} < \dfrac{1}{|g(x)|} < \dfrac{1}{|M|/2}$, that is $\dfrac{2}{3|M|} < \dfrac{1}{|g(x)|} < \dfrac{2}{|M|}$

Conclusion: $\left|\dfrac{1}{g(x)} - \dfrac{1}{M}\right| = |g(x)-M| \dfrac{1}{|M|}\dfrac{1}{|g(x)|} < |g(x)-M| \dfrac{2}{|M|^2}$

if $0<|x-a|<\delta_1$.

- To finish, pick $\varepsilon'' = \dfrac{\varepsilon}{2}|M|^2 > 0$ because $|M|>0$

Since $\lim_{x\to a} g(x) = M$, we can find $\delta_2 > 0$ so that if

$0<|x-a|<\delta_2$, then $|g(x) - M| < \varepsilon'' = \dfrac{\varepsilon}{2}|M|^2$

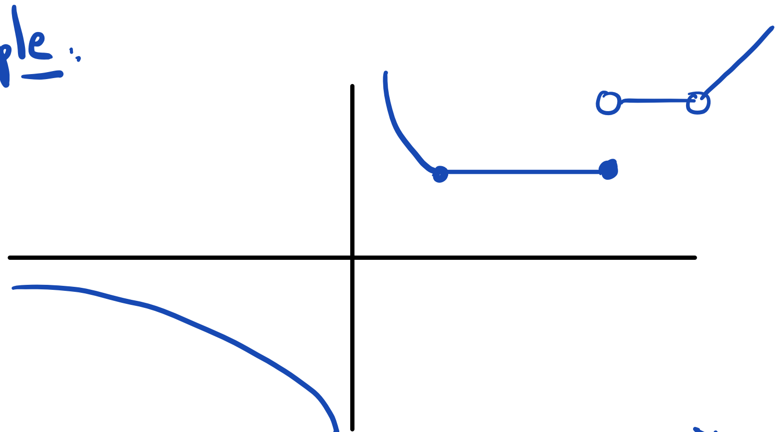
Pick $\boxed{\delta = \min\{\delta_1, \delta_2\} > 0}$. We get

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| \underset{\delta\le\delta_1}{<} |g(x)-M| \frac{2}{|M|^2} \underset{\delta\le\delta_2}{<} \varepsilon'' \frac{2}{|M|^2} = \frac{\varepsilon}{2}|M|^2 \cdot \frac{2}{|M|^2} = \varepsilon$$

as long as $0<|x-a|<\delta$.

§2 Continuous functions

Example:



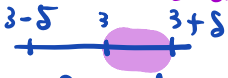
$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x<0 \\ \frac{1}{x} & \text{if } 0<x<1 \\ 1 & \text{if } 1\le x\le 3 \\ 2 & \text{if } 3<x<4 \\ x-2 & \text{if } x>4 \end{cases}$$

Domain of $f = \mathbb{R}\setminus\{0,4\}$
$= \{x : x\neq 0, 4\}$

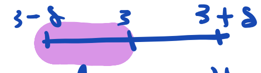
- f is continuous everywhere in D except at $x=3$, because

$\lim_{x\to 3^+} f(x) = 2 \neq f(3)$ & $\lim_{x\to 3^-} f(x) = 1 = f(x)$, so no limit at 3 exists!

In ε/δ-definition use $3<x<x+\delta$ side



use $3-\delta<x<3$ side



- We can define f at $x=4$ (that is "extend the function f to $x=4$") in a continuous fashion because $\lim_{x\to 4} f(x) = 2$. So declaring $f(4)=2$ will make f continuous at $x=4$

- We cannot extend f to $x=0$ in a continuous way because $\lim_{x\to 0} f(x)$ does not exist.

<u>Definition</u> A function f defined in a neighborhood of a point $x=a$ is <u>continuous at a</u> if:

(1) $f(x)$ is defined at $x=a$

(2) $\lim_{x\to a} f(x)$ exists and it equals $f(a)$.

<u>Definition</u>: We say f is <u>continuous</u> if it's continuous everywhere in its domain.

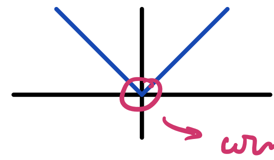
<u>Note</u>: Saying $\lim_{x\to a} f(x) = f(a)$ is the <u>same</u> as saying $\lim_{x\to a} (f(x) - f(a)) = 0$. Using increments, we can write

$$\lim_{\Delta x\to 0} f(a+\Delta x) - f(a) = \lim_{\Delta x\to 0} \Delta f = 0.$$

<u>Key Proposition</u>: If $f'(a)$ exists, then f MUST be continuous at a

Why? $\lim_{\Delta x\to 0} \Delta f = \lim_{\Delta x\to 0} \underbrace{\frac{\Delta f}{\Delta x}}_{\to f'(a)} \underbrace{\Delta x}_{\to 0} \underset{\text{Product Rule}}{=} f'(a)\cdot 0 = 0.$

⚠ f can be continuous at $x=0$ without having $f'(a)$



Example $f(x) = |x|$ & $a=0$

Clear: $\lim_{x\to 0} |x| = 0 = |0| = f(0)$ because

$\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0$ & $\lim_{x\to 0^-} |x| = \lim_{x\to 0^-} -x = -0 = 0.$

Aside: $|b+c| \le |b| + |c|$ for any b, c real numbers.

We show this by a case-by-case analysis:

(1) If $b, c \ge 0$, then $|b+c| = b+c = |b|+|c| \le |b|+|c|$

(2) If $b, c \le 0$ — $|b+c| = -b-c = |b|+|c| \le |b|+|c|$

(3) If $b \ge 0$, $c \le 0$ & $b+c \ge 0$:

$$|b+c| = b+c = |b|+c = |b|-|c| \le |b|+|c|$$

(same idea works if $b \le 0$, $c \ge 0$ & $b+c \ge 0$)

(4) If $b \ge 0$, $c \le 0$ & $b+c \le 0$

$$|b+c| = -b-c = -|b|+|c| \le |b|+|c|$$

(same idea works if $b \le 0$, $c \ge 0$ & $b+c \le 0$)

Consequence: $|b-c| = |b+(-c)| \le |b| + |-c| = |b|+|c|$