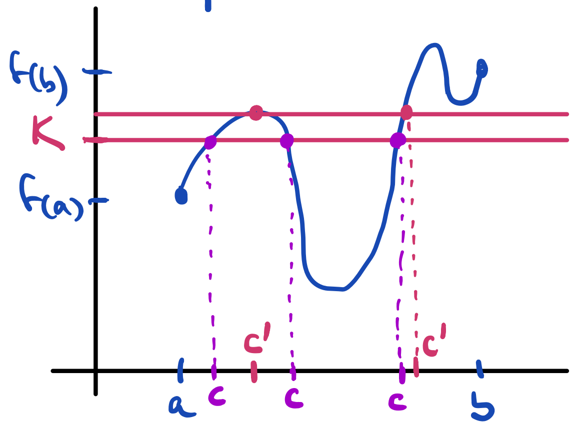
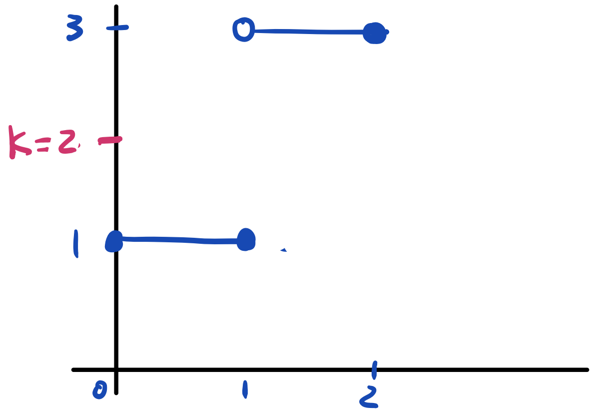
Lecture VII: §2.6 (cont) The Intermediate, Mean & Extreme Value Theorems

§1 The Intermediate Value Theorem (IVT):

IVT: If $f: \overset{=\{x\,:\,a\le x\le b\}}{[a,b]} \longrightarrow \mathbb{R}$ is continuous, then every number K in between $f(a)$ & $f(b)$ is attained, meaning we can find c in $[a,b]$ with $f(c)=K$.



vs



The graph crosses the horizontal line $y=K$ at least once for any K in between $f(a)$ & $f(b)$

We have 3 choices for c with $f(c)=K$. & 2 choices for c' with $f(c')=K'$.

$K=2$ is in between $f(0)=1$ & $f(2)=3$ but it's NOT attained

Special case: If $f(a)>0$ & $f(b)<0$ (or viceversa) & f is continuous then we can find c in $[a,b]$ with $f(c)=0$ (pick $K=0$)

Q: How can we use this result?

Example: $f(t) = 3t^2+\underbrace{t^3}_{\text{this is the dominant term}}+1$ continuous, $\lim_{t\to\infty} f(t)=+\infty$

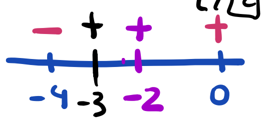
$\lim_{t\to-\infty} f(t)=-\infty$

Claim: $f(t)$ has a real root. Meaning we can find t_0 with $f(t_0)=0$ Q Can we estimate t_0? A YES

- $f(0) = 0+0+1 = 1>0$, $f(-4) = 3\cdot 16-64+1 = -15<0$

So we have a root in between 0 & -4.

- Better approximation? Bisect the interval.



→ $f(-2) = 3.4 - 8 + 1 = 5 > 0$ ⇝ a real root in between -4 & -2.

→ Once again $f(-3) = 1 > 0$ ⇝ ———————— -4 & -3

We can keep going like this $f(-3.5) = ?$, ...

(This is Newton's method for solving equations)

§2 The Mean Value Theorem (MVT)

MVT: Assume that $f: [a,b] \longrightarrow \mathbb{R}$ satisfies

(1) f is continuous on $[a,b]$ (meaning $f(a) = \lim_{x\to a^+} f(x)$, $f(b) = \lim_{x\to b^-} f(x)$ & f is continuous at each x with $a<x<b$)

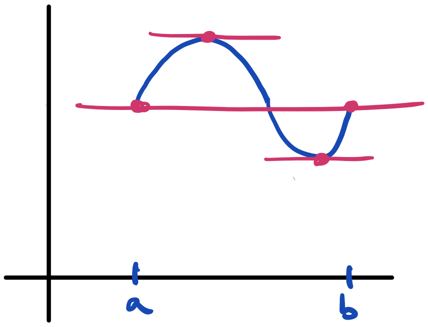
(2) f is differentiable on (a,b) ($f'(x)$ exists for each x with $a<x<b$)

Then, we can find c in (a,b) where $f'(c) = \frac{f(b)-f(a)}{b-a}$.

slope of the tangent line at $(c, f(c))$

slope of the secant line through $(a, f(a))$ & $(b, f(b))$

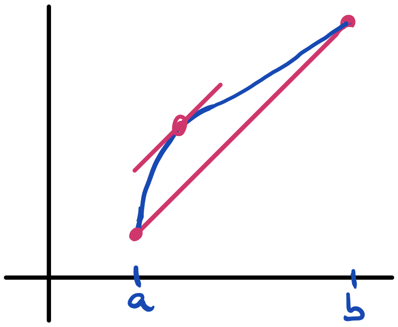
Pictorial argument:



$f(a) = f(b)$

2 values for c

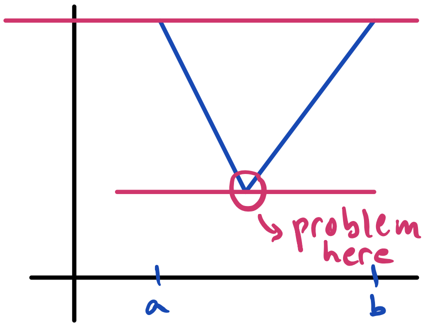
["Rolle's Thm"]



$f(a) \neq f(b)$

1 value for c

["horizontal line if you tilt your head"]



NON-example

$f(a) = f(b)$

Don't have a horiz. tangent line

A proof will be discussed in Appendix A4 (future lecture)

Application: We can use the sign of f' to predict the growth behavior of f, assuming f is continuous on $[a,b]$ & differentiable on (a,b).

Sign of f' on (a,b)	$+$	$-$	0
Growth behavior of f	strictly INCREASING	strictly DECREASING	constant

Why? Pick s, t with $a < s < t < b$. So f is continuous on $[s,t]$ & differentiable on (a,b)

By MVT we can find c in (s,t) with

$$f'(c) = \frac{f(t) - f(s)}{t-s}$$

So $f'(c)$ & $(f(t) - f(s))$ have the same sign.

(1) If $f'(c) > 0$, then $f(t) > f(s)$ $\rightsquigarrow$ f is strictly INCR.

(2) If $f'(c) < 0$, then $f(t) < f(s)$ $\rightsquigarrow$ f ______ DECR

(3) If $f'(c) = 0$ ___ $f(t) = f(s)$ $\rightsquigarrow$ f is constant

↳ The choice of $s<t$ was arbitrary!

§3. The Extreme Value Theorem (EVT):

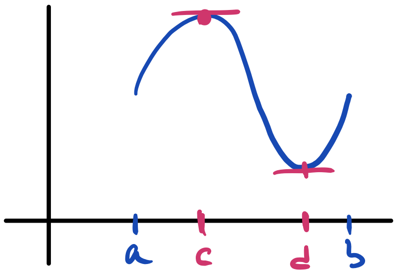
EVT: If f is continuous on $[a,b]$, then f attains both a maximum & a minimum value in $[a,b]$ ("the extreme values")

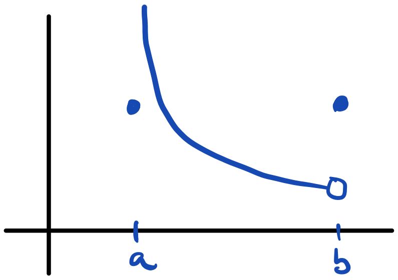
Why? ① Need to show f is bounded (i.e., we can find M & N with $M \leq f(x) \leq N$ for all x with $a \leq x \leq b$

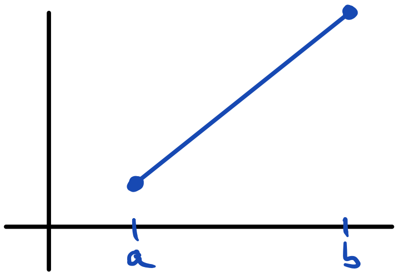
② Can adjust M & N to be the least lower & upper bounds

[This is done in Appendix A3 and it uses properties of the real numbers]

③ Show these optimal bounds M & N are achieved.







f(c) MAX
f(d) MIN
f differentiable at (a,b)
& $f'(c) = f'(d) = 0$

$\lim_{x\to a^-} f(x) = +\infty$

- no min attained
- no max ———

$f: [0,2] \longrightarrow \mathbb{R}$
not continuous at $x = 0$
$x = 2$

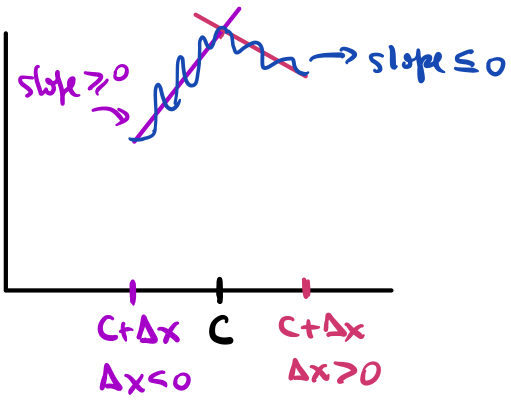
f(a) MIN
f(b) MAX

- f is differentiable but not at a or b.
- f is continuous on [a,b]

⚠ It is important to consider the points a & b.
If f is differentiable on (a,b), the search for M & N is simpler.

Consequence: If c in (a,b) realizes an extreme value for f & f is continuous on [a,b] & differentiable at c, then $f'(c)=0$ (in short, the tangent line to the graph of f at $(c,f(c))$ is horizontal)

Why? Assume f(c) is a MAX value (if it's a MIN value, the argument is similar).



- Use slope of secants through $(c,f(c))$ & $(c+\Delta x, f(c+\Delta x))$ $\xrightarrow[\Delta x\to 0]{}$ slope of tangent line at $(c, f(c))$
- To the right: slope of secant is ≤ 0
- To the left: ——— ≥ 0
- So the sign in the limit is = 0, that is, $f'(c)=0$.

Problem 1: Show that $f_{(x)} = 4+x^2$ has max & min values on $[-2,2]$ & find them. What happens if we restrict to $(-2,2)$?

Solution: f is continuous on $[-2,2]$ By EVT, we have a max & a min value on $[-2,2]$. Since f is differentiable on $(-2,2)$, we can find these values as follows:

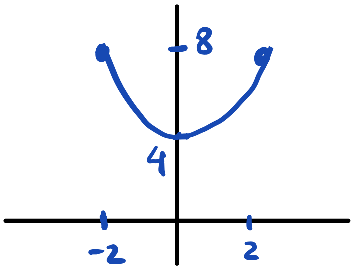
(1) Find x with $f'_{(x)}=0$ ($2x=0$, so $x=0$) ($f_{(0)}=4$)

(2) Compute $f_{(-2)}$ & $f_{(2)}$ ($f_{(-2)}=f_{(2)}=8$)

(3) Compare $f(-2)$ & $f(2)$ with the values from (1), pick the largest & smallest.

⇝ MAX $= 8$ (at $x = 2$ & -2)

MIN $= 4$ (at $x=0$)



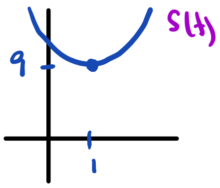
When restricted to $(-2,2)$ we still have a MIN value but we don't have a MAX since $\lim_{x\to 2^-} f_{(x)} = \lim_{x\to -2^+} f_{(x)} = 8$, f is str. increasing near $x=2$

——— decreasing ——— $x=-2$

Problem 2: Assume $s(t)=12-6t+3t^2$ is the position at time t of an object moving on a straight line. Find the velocity & acceleration & decide if at any point, the object changes direction.

Solution: $v(t)=s'(t) = -6+6t = 6(t-1)$

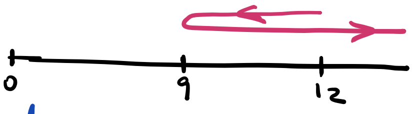
$a(t)=v'(t) = 6$ ⇝ constant!



To change direction means $v(t)$ must change sign!

Answer: At $t=1$ → vertex of the parabola it changes direction

$s(1)=9$



If $t<1$ $v(t)<0$ so $s(t)$ is strictly decreasing

If $t>1$ $v(t)>0$ so $s(t)$ is strictly increasing