Lecture VIII. §3.1 Derivatives of polynomials
§3.2 Product & Quotient Rules

§1 Derivation of polynomials:

Pick $f_{(x)} = c_n x^{n} + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 \rightarrow$ polynomial of degree n with real coeffs

$c_n, c_{n-1}, \ldots, c_1, c_0$ are fixed real numbers

Proposition: (1) $\frac{d}{dx} c = 0$ (derivative of a constant function)

(2) $\frac{d}{dx} x^n = n x^{n-1}$ for any positive integer n $(=1,2,3,\ldots)$

Why? Use the definition

(1) Set $c_{(x)} = c$ so $\frac{dc}{dx} = \lim_{\Delta x \to 0} \frac{\Delta c}{\Delta x} \overset{c(x+\Delta x) = c(x) = c}{=} \lim_{\Delta x \to 0} \frac{0}{\Delta x} = 0$

(2) Want to show: $\lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} = n x^{n-1}$.

To do so, we need to rewrite the numerator in a better way to "cancel" Δx.

Use $(a+b)^n = \underbrace{(a+b)(a+b)\cdots(a+b)}_{n \text{ times}}$ & distribute. We get

Binomial Theorem:

$$(a+b)^n = a^n + n a^{n-1} b + \frac{n(n-1)}{2} a^{n-2} b^2 + \cdots$$

$$\cdots + \boxed{\frac{n(n-1)\cdots(n-k)}{1\cdot 2 \cdots k}} a^{n-k} b^k + \cdots$$

$$\cdots + \frac{n(n-1)}{2} a^2 b^{n-2} + n a b^{n-1} + b^n$$

$$= a^n + n a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots + \boxed{\binom{n}{k}} a^{n-k} b^k + \cdots + \binom{n}{2} a^2 b^{n-2} + n a b^{n-1} + b^n$$

↳ "n choose k" (entries of Pascal Δ)

(choose the k terms among the total of n factors where we pick b)

This expression is palindromic in a & b & it has integer coefficients

Example $(a+b)^2 = a^2 + \underset{\binom{2}{1}}{2}ab + b^2$, $(a+b)^3 = a^3 + \underset{\binom{3}{1}}{3}a^2 b + \underset{\binom{3}{2}}{3} a b^2 + b^3$, etc.

In our case: $(x+\Delta x)^n = x^n + n x^{n-1}\Delta x + \cdots + \binom{n}{k} x^{n-k}(\Delta x)^k + \cdots + n x (\Delta x)^{n-1} + (\Delta x)^n$

So $\dfrac{(x+\Delta x)^n - x^n}{\Delta x} = n x^{n-1}\dfrac{\cancel{\Delta x}}{\cancel{\Delta x}} + \binom{n}{2} x^{n-2}\dfrac{(\Delta x)^{\cancel{2}}}{\cancel{\Delta x}} + \cdots \binom{n}{k} x^{n-k}\dfrac{(\Delta x)^{\cancel{k}\, k-1}}{\cancel{\Delta x}} + \cdots + \dfrac{(\Delta x)^{\cancel{n}\, n-1}}{\cancel{\Delta x}}$

$\downarrow$ $\Delta x \neq 0$

Each summand $\binom{n}{k} x^{n-k}(\Delta x)^{k-1} \xrightarrow[\Delta x]{} 0$ if $k>1$, so only the first one survives when taking $\lim\limits_{\Delta x \to 0}$. We get $\dfrac{d x^n}{dx} = n x^{n-1}$ ☺.

Key Properties: (1) $\dfrac{d}{dx}(c f(x)) = c \dfrac{df}{dx}$

(2) $\dfrac{d}{dx}(f(x) + g(x)) = \dfrac{df}{dx} + \dfrac{dg}{dx}$

Why? Use Product rule for limits & sum rule for limits.

Important Consequence If $f = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$ is a polynomial of degree $n \geq 0$, then $\dfrac{df}{dx} = c_n n x^{n-1} + c_{n-1}(n-1) x^{n-1} + \cdots + c_2 2x + c_1$.

Q: How to interpret the 2 key properties?

A: "Differentiation is a linear operator in the space of functions"

$f \longmapsto f' =$ new function

Linear means:
- it sends sums to sums (this is property (2))
- it sends multiplication by constants (called scalars) to multiplication by constants (this is property (1))

Linear properties & spaces / maps satisfying them are the subject of Linear Algebra (MATH 2568).

§2. Product Rule

Theorem 1 Assume that $f(x)$ & $g(x)$ are differentiable. Then, the product function $h(x) = f(x)\,g(x)$ is also differentiable. Furthermore

$$\frac{d}{dx}(f(x)g(x)) = \frac{df}{dx}\cdot g(x) + f(x)\frac{dg}{dx}.$$

Q: Why is the formula valid?

A: Use the definition!

$$\frac{f(x+\Delta x)\,g(x+\Delta x)-f(x)\,g(x)}{\Delta x} \overset{+\,\&\,-\,f(x)g(x+\Delta x)\ (\text{"mixed Term"})}{=} \frac{f(x+\Delta x)\,g(x+\Delta x)-f(x)\,g(x+\Delta x)}{\Delta x} + \frac{f(x)\,g(x+\Delta x)-f(x)\,g(x)}{\Delta x} =$$

$$\underset{\text{rearrange}}{=} \boxed{\frac{f(x+\Delta x)-f(x)}{\Delta x}}\, g(x+\Delta x) + f(x)\,\boxed{\frac{g(x+\Delta x)-g(x)}{\Delta x}}$$

$$\boxed{\frac{f(x+\Delta x)-f(x)}{\Delta x}} \xrightarrow{\Delta x\to 0} \frac{df}{dx}, \qquad g(x+\Delta x) \xrightarrow{\Delta x\to 0} g(x)\ \text{(g differentiable, so continuous)}, \qquad \boxed{\frac{g(x+\Delta x)-g(x)}{\Delta x}} \xrightarrow{\Delta x\to 0} \frac{dg}{dx}$$

So
$$\frac{d(fg)}{dx} = \lim_{\Delta x\to 0}\frac{\Delta(fg)}{\Delta x} = \frac{df}{dx}\,g(x) + f(x)\,\frac{dg}{dx}$$
!! :)

EXAMPLE 1: For $f=g=x$, we get $\frac{d\,x^2}{dx} = 1\cdot x + x\cdot 1 = 2x$

For $f = x^{n-1}$, $g(x)=x$, we get $\frac{d\,x^n}{dx} = \frac{d\,(x^{n-1}x)}{dx} = \frac{d\,x^{n-1}}{dx}\cdot x + x^{n-1}\cdot 1$

Now if we know $\frac{d\,x^{n-1}}{dx} = (n-1)\,x^{n-2}$, we get

$$\frac{dx^n}{dx} = (n-1)\,x^{n-2}\cdot x + x^{n-1} = (n-1)\,x^{n-1} + x^{n-1} = n\,x^{n-1}.$$

(This is an example of an argument by induction: we know it to be true for $n=1$ & we see that the result being valid for $n-1$ forces it to be valid for n. Thing of dominoes stacked together. The first one falls and each dominoe pushes the next one, so in the end they all fall.)

- We see that the product rule forces the power rule.

EXAMPLE 2: $f = (2x-5)(x^3-4x+8)$

- Use distribution $f = 2x^4 - 5x^3 - 8x^2 + 36x - 40 \rightsquigarrow \frac{df}{dx} = 8x^3 - 15x^2 - 16x + 36.$
- Use Product Rule: $\frac{df}{dx} = 2(x^3-4x+8) + (2x-5)(3x^2-4) = 8x^3 - 15x^2 - 16x + 36$

§3 Quotient Rule:

Q: Given $f(x)$ & $g(x)$, when can we define $h(x) = \frac{f(x)}{g(x)}$?

A: Need $g(x) \neq 0$, so the domain of h becomes:

Domain of $h(x) = \{x : x$ in the domain of both f & g, & $g(x) \neq 0\}$

Q: How to differentiate h, assuming f & g are differentiable?

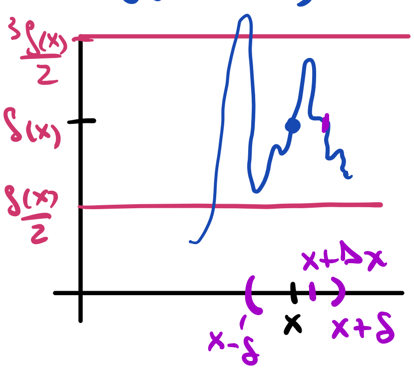
A: Write $h(x) = f(x) \cdot \frac{1}{g(x)}$ & use product rule:

$$\frac{dh}{dx} = \frac{df}{dx} \cdot \frac{1}{g(x)} + f \frac{d}{dx}\left(\frac{1}{g(x)}\right).$$

All we need to do is to find a formula for $\frac{d}{dx}\left(\frac{1}{g(x)}\right)$.

Theorem 2: Assume $g(x) \neq 0$ & g is differentiable at x. Then $p(x) = \frac{1}{g(x)}$ is defined in a neighborhood of x and it is also differentiable at x, with
$$\frac{dp}{dx} = -\left(\frac{dg}{dx}\right) \Big/ g^2(x) = \frac{-g'}{g^2}.$$

Why? g is differentiable at x so it's continuous at x.
In particular, if $g(x) \neq 0$, we can find $\delta > 0$ so that $g(x+\Delta x)$ & $g(x)$ have the same sign if $|\Delta x| < \delta$.



Indeed:
Say $g(x) > 0$. Pick $\varepsilon = \frac{g(x)}{2} > 0$. We can find $\delta > 0$ so that $|g(x+\Delta x) - g(x)| < \varepsilon$

$$0 < \frac{g(x)}{2} = g(x) - \varepsilon < g(x+\Delta x) < g(x) + \varepsilon = \frac{3g(x)}{2}$$

If $g(x) < 0$, the argument is similar.

Since $g(x+\Delta x)$ has the same sign as $g(x)$ if $|\Delta x| < \delta$, it's never 0, so we can define $p(x+\Delta x)$ for each $|\Delta x| < \delta$.
It remains to show p is differentiable at x. We use the definition

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(\frac{1}{g(x+\Delta x)} - \frac{1}{g(x)}\right)$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \frac{g(x) - g(x+\Delta x)}{g(x)\, g(x+\Delta x)}$$

$$\underset{\text{reorder}}{=} \lim_{\Delta x \to 0} - \boxed{\frac{g(x+\Delta x) - g(x)}{\Delta x}} \frac{1}{g(x)} \boxed{\frac{1}{g(x+\Delta x)}} \overset{\text{Limit Laws.}}{=} \frac{-g'(x)}{g^2(x)}$$

$$\boxed{\frac{g(x+\Delta x) - g(x)}{\Delta x}} \xrightarrow{\Delta x \to 0} \frac{dg}{dx} \qquad \boxed{\frac{1}{g(x+\Delta x)}} \xrightarrow{\Delta x \to 0} \frac{1}{g(x)} \quad (g \text{ is continuous})$$

Consequences:

① We get the power rule for negative exponents!

Ex: $f(x) = x^{-3} = \frac{1}{x^3} \rightsquigarrow f'(x) = \frac{-g'}{g^2} = \frac{-3x^2}{x^6} = \frac{-3}{x^4}$.

In general: $f(x) = x^{-n} = \frac{1}{x^n} \rightsquigarrow f'(x) = \frac{-n x^{n-1}}{(x^n)^2} = \frac{-n x^{n-1}}{x^{2n}} = -n x^{n-1-2n}$

$$= -n x^{-1-n} = \frac{-n}{x^{n+1}} = -n x^{-n-1}!$$

② **Quotient Rule:** $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{df}{dx}\frac{1}{g} + f \frac{d}{dx}\left(\frac{1}{g}\right)$

$$= \frac{f'}{g} + f\left(\frac{-g'}{g^2}\right) = \frac{f'g - fg'}{g^2}.$$

Problem: Decide where $f = \frac{x+1}{x-1}$ is defined,
- continuous,
- differentiable.

Solution: We only need to check when $x-1 \neq 0$ since the numerator & denominator of f are differentiable functions.

So. Domain of $f = (x \neq 1)$ $\rightsquigarrow$ cannot extend continuously to $x=1$ because $\lim_{x\to 1^+} f(x) = \infty$ | $\lim_{x \to 1^-} f(x) = -\infty$

- f is continuous & differentiable on it's domain

$$\frac{df}{dx} = \frac{1\cdot(x-1) - (x+1)\cdot 1}{(x-1)^2} = \frac{x-1-(x+1)}{(x-1)^2} = \frac{-2}{(x-1)^2}.$$

Remark: f is what's called a rational function (ratio of polynomials)