Lecture X : § 3.4 Some Trigonometric functions

So far, we have derivation rules for

(1) addition $(f_{(x)}+g_{(x)})' = f'_{(x)} + g'_{(x)}$

(2) scalar multiplication : $(c f_{(x)})' = c f'_{(x)}$ (c fixed real number)

(3) product $(f_{(x)} \cdot g_{(x)})' = f'_{(x)} g_{(x)} + f_{(x)} g'_{(x)}$

(4) quotient $\left(\frac{f_{(x)}}{g_{(x)}}\right)' = (f'(x) g(x) - f(x) g'(x)) / g^2_{(x)}$.

(5) composition $\rightsquigarrow$ Chain Rule $(g \circ f)'_{(x)} = g'(f_{(x)}) \cdot f'_{(x)}$.

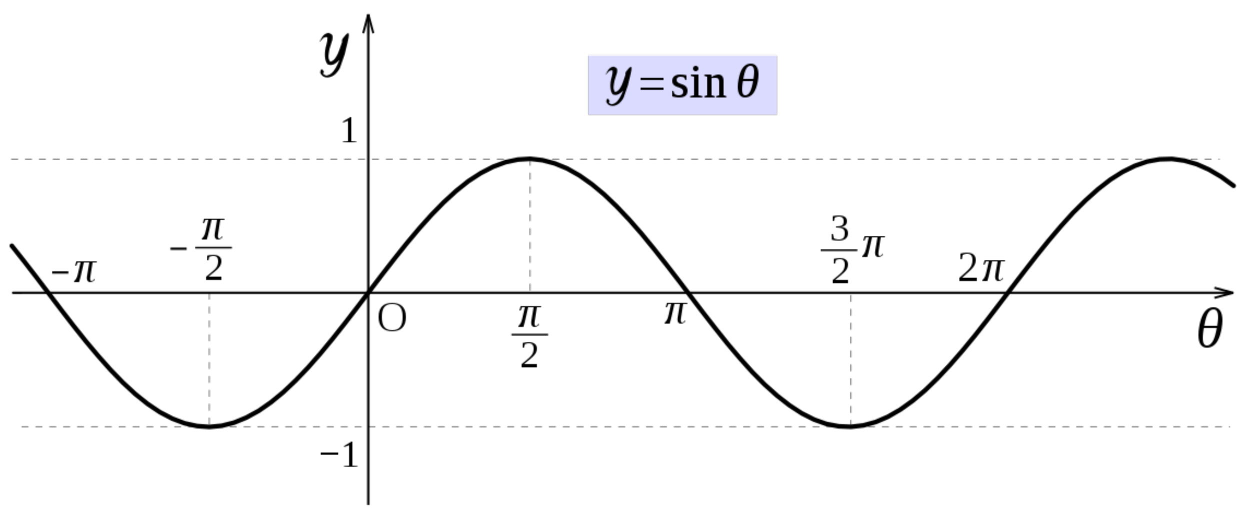
Combining these, we get explicit formulas for derivatives of polynomials, rational functions & powers.

Q : What about other functions? Can we have more building blocks than x^n?

§1. Trigonometric functions:

2 Basic Trig. functions : $\sin x$ & $\cos x$ $\left(\tan x = \dfrac{\sin x}{\cos x}\right)$

① $\boxed{\sin x}$

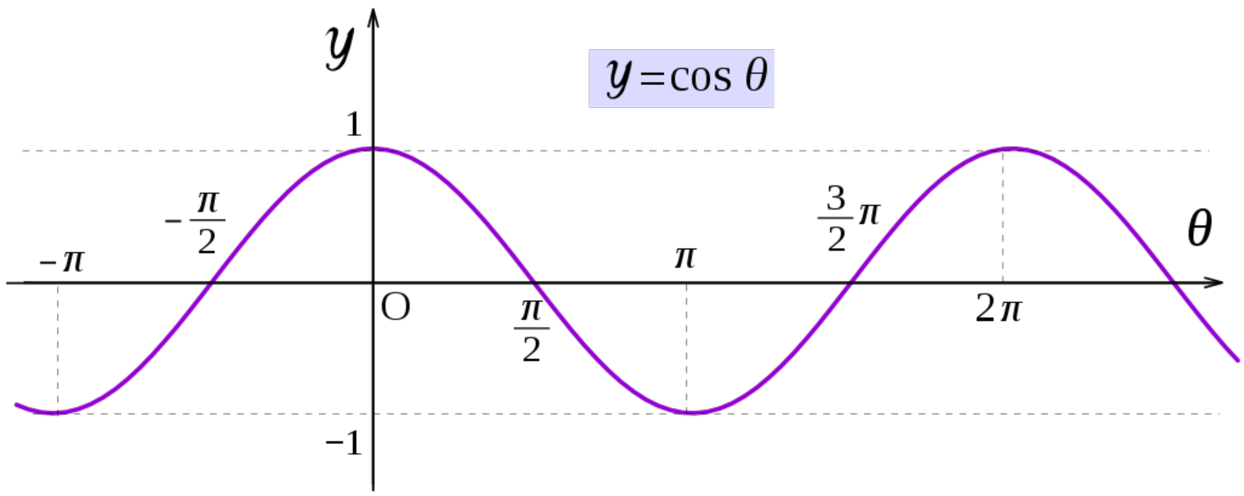


Source : https://upload.wikimedia.org/wikipedia/commons/8/87/Y%3Dcos%28theta%29.svg

Properties:
- $\sin(x)$ is periodic, with period 2π. This means that $\sin(x+2\pi) = \sin(x)$
- $\sin x$ is ODD, meaning $\sin(-x) = -\sin x$.
- $-1 \le \sin x \le 1$ for all x

Useful values:
- $\sin(0) = \text{sen}(\pi) = 0$
- $\sin\left(\frac{\pi}{2}\right) = 1$, $\text{sen}\left(\frac{3\pi}{2}\right) = -1$
- $\text{sen}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, $\text{sen}\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$, $\text{sen}\left(\frac{\pi}{6}\right) = \frac{1}{2}$

② $\boxed{\cos x}$



Source: https://upload.wikimedia.org/wikipedia/commons/8/87/Y%3Dcos%28theta%29.svg

Properties:
- $\cos(x)$ is periodic, with period 2π. This means that $\cos(x+2\pi) = \cos(x)$
- $\cos x$ is EVEN, meaning $\cos(-x) = \cos x$.
- $-1 \leq \cos x \leq 1$ for all x

Useful values:
- $\cos(0) = 1$, $\cos(\pi) = -1$
- $\cos\left(\frac{\pi}{2}\right) = \cos\left(\frac{3\pi}{2}\right) = 0$
- $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$, $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

Graphs suggest a shift. Indeed, we have

$$\boxed{\cos x = \text{sen}\left(x + \frac{\pi}{2}\right)}$$

These 2 functions vary nice and smoothly, so they should be differentiable. We'll find the formulas via limits, computing the increments. For this, we'll need additive formulas for $\sin x$ & $\cos x$.

(1) $\sin(a \pm b) = \sin a \cos b \pm \cos a \,\text{sen}\, b$ (mixed)

(2) $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$ (unmixed)

Fundamental Equation $\quad \sin^2 x + \cos^2 x = 1$

Next, we compute the derivatives of $\sin x$ & $\cos x$ via increments:

Theorem (1) $\frac{d}{dx}\sin x = \cos x$ $\qquad$ (2) $\frac{d}{dx}\cos x = -\sin x$

Q Why? A Use the method of increments!

(1)
$$\frac{d}{dx}\sin x = \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin x}{\Delta x}$$

Additive formula (1)
$$= \lim_{\Delta x \to 0} \frac{\sin x \cos(\Delta x) + \cos x \sin(\Delta x) - \sin x}{\Delta x}$$

rearrange
$$= \lim_{\Delta x \to 0} \sin x \underbrace{\boxed{\left(\frac{\cos(\Delta x) - 1}{\Delta x}\right)}}_{\text{NEEDS MORE WORK } (\frac{0}{0} \text{ indeterminacy})} + \cos x \underbrace{\boxed{\frac{\sin(\Delta x)}{\Delta x}}}_{\downarrow \Delta x \to 0 \;\; 1}$$

Claim:
$$\lim_{\Delta x \to 0} \frac{\cos(\Delta x) - 1}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cos(\Delta x) - 1}{\Delta x} \, \frac{\cos(\Delta x) + 1}{\cos(\Delta x) + 1}$$
$$= \lim_{\Delta x \to 0} \frac{\cos^2(\Delta x) - 1}{\Delta x(\cos \Delta x + 1)} = \lim_{\Delta x \to 0} \frac{\sin^2 \Delta x}{\Delta x(\cos \Delta x + 1)}$$
$$= \lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} \, \frac{\sin \Delta x}{\cos \Delta x + 1} = 1 \cdot \frac{0}{2} = 0$$

Conclusion: $\frac{d}{dx}\sin x = \sin x \cdot 0 + \cos x \cdot 1 = \cos x$. ✓

(2)
$$\frac{d}{dx}\cos x = \lim_{\Delta x \to 0} \frac{\cos(x+\Delta x) - \cos x}{\Delta x}$$

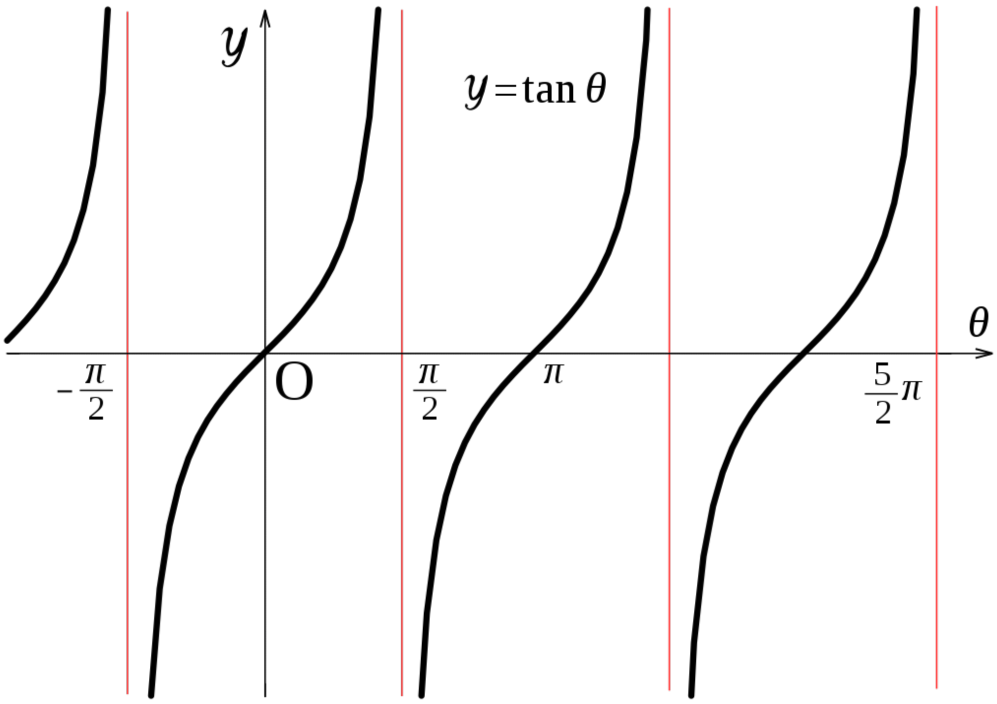
Additive formula (2)
$$= \lim_{\Delta x \to 0} \frac{\cos x \cos(\Delta x) - \sin x \sin(\Delta x) - \cos x}{\Delta x}$$

rearrange
$$= \lim_{\Delta x \to 0} \cos x \underbrace{\boxed{\left(\frac{\cos \Delta x - 1}{\Delta x}\right)}}_{\downarrow \Delta x \to 0 \;\; 0} - \sin x \underbrace{\boxed{\frac{\sin(\Delta x)}{\Delta x}}}_{\downarrow \Delta x \to 0 \;\; 1} = \cos x \cdot 0 - \sin x \cdot 1 = -\sin x$$ ✓

§2. Other Trigonometric Functions:

③ $\tan x = \dfrac{\sin x}{\cos x}$ This is defined whenever $\cos x \neq 0$, that is

$x \neq \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$
$\quad -\frac{\pi}{2}, -\frac{3\pi}{2}, -\frac{5\pi}{2}, \dots$ so $\boxed{x \neq \frac{(2k+1)\pi}{2} \text{ for } k \text{ integer}}$



Source: https://commons.wikimedia.org/wiki/File:Y%3Dtan(x).svg

$\lim\limits_{x \to \frac{\pi}{2}^-} \dfrac{\sin x}{\cos x} = +\infty \quad \left(\frac{1}{0^+}\text{-type}\right)$, $\lim\limits_{x \to \frac{\pi}{2}^+} \dfrac{\sin x}{\cos x} = -\infty \left(\frac{1}{0^-}\text{-type}\right)$

Properties:
- $\tan x$ is Periodic, with period π, that is $\tan(x+\pi) = \tan x$
- $\tan x$ is ODD, so $\tan(-x) = -\tan x$
- Image (or Range) of Tan is all $\mathbb{R}$

To compute $\frac{d}{dx}\tan x$, we use the Quotient Rule

$$\frac{d}{dx}\tan x = \frac{(\sin x)'\cos x - \sin x\,(\cos x)'}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

⇝ The domain of $\frac{d}{dx}\tan x$ and $\tan x$ is the same!

④ $\text{cotan}(x) = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$ Defined when $\sin x \neq 0$, that is $x \neq 0, \pi, 2\pi, \ldots$ $-\pi, -2\pi, \ldots$ so $\boxed{x \neq k\pi \text{ for } k \text{ integer}}$

Next, we compute its derivative with the power + chain rule:

$$\frac{d}{dx}\text{cotan}(x) = \frac{-1}{\tan^2 x} \cdot \frac{d}{dx}\tan x = -\frac{1}{\frac{\sin^2 x}{\cos^2 x}} \frac{1}{\cos^2 x} = \frac{-1}{\sin^2 x}$$

It has the same domain as $\text{cotan}\, x$

⑤ $\sec(x) = \frac{1}{\cos x}$ Domain = Domain of $\tan x$

$$\frac{d}{dx}\sec(x) = \frac{-1}{\cos^2 x} \cdot \frac{d}{dx}\cos x = \frac{-1}{\cos^2 x}(-\sin x) = \frac{\sin x}{\cos^2 x}$$

⑥ $\csc(x) = \frac{1}{\sin x}$ Domain = Domain of $\text{cotan}\, x$

$$\frac{d}{dx}\csc(x) = \frac{-1}{\sin^2 x}\frac{d}{dx}\sin x = \frac{-1}{\sin^2 x}\cos x = \frac{-\cos x}{\sin^2 x}$$

⚠ We don't need to memorize these formulas, we can just use the Quotient Rule to get them from the basic building blocks = $\sin x$ & $\cos x$.

§3 Examples:

(1) $h(x) = \sin(x^3) \rightsquigarrow h' = ?$

Use Chain Rule: $g(y) = \sin y \rightsquigarrow g' = \cos y$ $h = g \circ f$.

$f(x) = x^3 \rightsquigarrow f' = 3x^2$

So $h' = g'(f(x)) \cdot f'(x) = \cos(x^3) \cdot 3x^2$.

(2) $h(x) = \sin^3 x$ $\qquad h = g \circ f$ with $g(y) = y^3 \rightsquigarrow g' = 3y^2$
$f(x) = \sin x \rightsquigarrow f' = \cos x$

So $h' = g'(f(x)) \cdot f' = 3(\sin^2 x) \cdot \cos x$.

(3) $f(x) = \sin^2 x \cos^3 x \rightsquigarrow f' = ?$

Use Product + Chain Rule.

$$
\begin{aligned}
f' &= (\sin^2 x)' \cos^3 x + \sin^2 x \cdot (\cos^3 x)' \\
&= 2 \sin x \cdot \cos x \cdot \cos^3 x + \sin^2 x \; 3\cos^2 x \, (-\sin x) \\
&= 2 \sin x \cos^4 x - 3 \sin^3 x \cos^2 x \\
&= \sin x \cos^2 x \, (2\cos^2 x - 3\sin^2 x) \\
&= \sin x \cos^2 x \, (2(1-\sin^2 x) - 3\sin^2 x) \\
&= \sin x \cos^2 x \, (2 - 5\sin^2 x)
\end{aligned}
$$

(4) $f(x) = \cos((4x+1)^2) \rightsquigarrow f' = ?$

Again, we use the Chain Rule:

$$
\begin{aligned}
f' &= -\sin((4x+1)^2) \cdot 2(4x+1) \cdot 4 \\
&= -\sin((4x+1)^2) \; 8 \, (4x+1).
\end{aligned}
$$