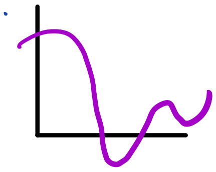
Lecture XI : §3.5 Implicit functions & fractional exponents

- So far, our functions were given as $f: D \to \mathbb{R}$ via explicit formulas

Example $y = f(x) = (x^3 + 4x)^{10}$ or $y = \sin x$

Here: y = dependent variable & x = independent variable

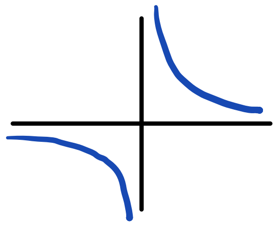
From this data we get a curve = the graph of the function



- Often times we deal with curves given by a relation between x & y, but we cannot solve for x or y. ⇝ implicit functions

§1 Examples: Classical plane curves

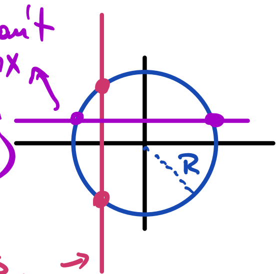
① Hyperbola $xy = 1$



graph of $y = \frac{1}{x}$ on $\mathbb{R} \setminus \{0\}$

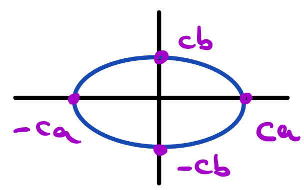
② Circle of radius R centred at 0

$$x^2 + y^2 = R^2$$



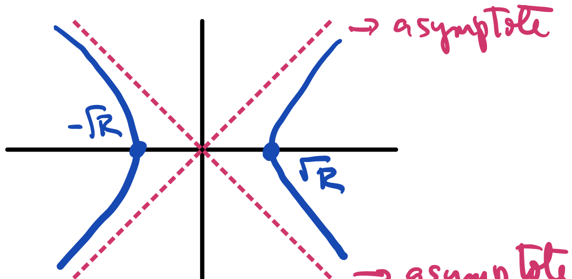
Note We can think of $x^2+y^2 = R^2$ as two half-circles (above & below the x-axis). We can solve for y on each half: $y = \pm\sqrt{R^2 - x^2}$ (defined for x in $[-R, R]$)

③ Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = c^2$ $a, b, c > 0$



(here $a > b$)

④ $x^2 - y^2 = R$ $(R > 0)$ is again a hyperbola



Factor the (LHS) to get

$$(x+y)(x-y) = R$$

asymptotes: $\begin{cases} x+y=0 \\ x-y=0 \end{cases}$

5 Equation $2y^2-2xy = 10-x^2 \rightsquigarrow 2y^2-2xy+(x^2-10)=0$

Solve for y with quadratic formula:

$$y = \frac{-(-2x) \pm \sqrt{4x^2-4\cdot 2\,(x^2-10)}}{2\cdot 2} = \frac{2x \pm \sqrt{80-4x^2}}{4} = \frac{x \pm \sqrt{20-x^2}}{2}$$

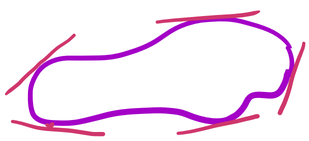
$|x| < \sqrt{20}$

Once again, there are 2 solutions, so we have 2 curves: the graphs of

$f_+ : [-\sqrt{20}, \sqrt{20}] \to \mathbb{R}$ & $f_- : [-\sqrt{20}, \sqrt{20}] \to \mathbb{R}$

$f_+(x) = \dfrac{x+\sqrt{20-x^2}}{2}$ $\qquad f_-(x) = \dfrac{x-\sqrt{20-x^2}}{2}$.

- Equations like this one can be "solved by radicals" only if the degree in y is ≤ 4 ["Galois Theory" explains why]

Obs:  Locally around a point, the curve is the graph of a function unless the tangent line at the point is vertical

INPUT: A curve and a point P where the tangent line at P is NOT vertical

GOAL Determine slope of the tangent line at P

Q: Can we do this WITHOUT solving for y, that is, without knowing the explicit formula for the function describing the curve near P)

A: YES!

§2. Implicit Differentiation:

Guiding Principles:

1. "If 2 functions are =, so are their derivatives"
2. Chain Rule & various derivation techniques can be used.
3. Check feasibility of operations to find bad point when the formula fails.

Back To examples:

(5) $2y^2 - 2xy = 10 - x^2$ Think $y = y(x)$

Take $\frac{d}{dx}$ on both sides using chain rule & product rule.

$$2(2y)\frac{dy}{dx} - 2\left(y + x\frac{dy}{dx}\right) = -2x$$

$$4y\,y' - 2y - 2x\,y' = -2x$$

Now, we solve for y':

$$(4y - 2x)\,y' - 2y = -2x$$

$$(4y - 2x)\,y' = -2x + 2y$$

$$\boxed{y' = \frac{-2x + 2y}{4y - 2x} = \frac{y - x}{2y - x}}$$

(*) only works if $2y - x \neq 0$.

Bad Points: $2y = x$

We go back to the original equation and see which points $P = (2y, y)$ satisfy the original equation (so they are pts on this curve)

$$2y^2 - 2(2y)y \stackrel{?}{=} 10 - (2y)^2$$

$$2y^2 - 4y^2 = 10 - 4y^2$$

$$2y^2 = 10$$

$$y = \pm\sqrt{5} \quad \rightsquigarrow \quad x = \pm 2\sqrt{5} = \pm\sqrt{20}$$

$\rightsquigarrow$ Problematic points: $(\sqrt{20}, \sqrt{5})$ & $(-\sqrt{20}, -\sqrt{5})$

Sanity check: since we have f_+ & f_- describing the curve, we can check our formula for y'.

- $f_+(x) = \dfrac{x + \sqrt{20 - x^2}}{2} \rightsquigarrow f_+' = \dfrac{1}{2} + \dfrac{1}{2}\dfrac{d}{dx}\left(\sqrt{20 - x^2}\right) \stackrel{\text{LATER}}{=} \dfrac{1}{2} + \dfrac{1}{4}\dfrac{-2x}{\sqrt{20 - x^2}}$

BUT $\sqrt{20-x^2} = 2y - x$ by the definition of f_+, so we get

$$f_+' = \frac{1}{2} - \frac{x}{2(2y-x)} = \frac{1}{2}\left(\frac{2y-x-x}{2y-x}\right) = \boxed{\frac{y-x}{2y-x}}$$ as we had earlier :)

- $f_-(x) = \dfrac{x - \sqrt{20-x^2}}{2}$. $\leadsto f_-'(x) = \frac{1}{2} - \frac{1}{2}\frac{d}{dx}\left(\sqrt{20-x^2}\right) = \frac{1}{2} - \frac{1}{4}\frac{(-2x)}{\sqrt{20-x^2}}$

BUT $\sqrt{20-x^2} = -2y + x$ by the definition of f_-, so we get

$$f_-' = \frac{1}{2} + \frac{x}{2(-2y+x)} = \frac{-2y+x+x}{2(-2y+x)} = \frac{-y+x}{-2y+x} = \boxed{\frac{y-x}{2y-x}} \checkmark$$

(2) $x^2+y^2 = R^2$ $\leadsto$ $y = y(x)$ gives $x^2 + y(x)^2 = R^2$

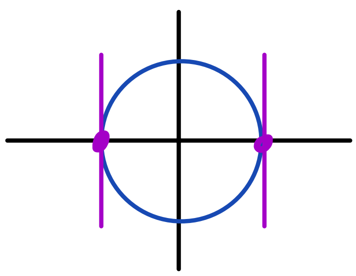
Take $\frac{d}{dx}$ on both sides: $2x + 2y \cdot \frac{dy}{dx} = 0$

$$2y\,y' = -2x$$

$$y' = \frac{-2x}{2y} \quad (*) \text{ issue when } y=0.$$

$$y' = -\frac{x}{y}$$

<u>Bad point</u>: $y=0$ so $x = \pm R$



Tangent lines are vertical, so we must swap x & y, ie think $x = x(y)$

$\leadsto 2x\,x' + 2y = 0$

$$x' = \frac{-2y}{2x}$$

$$x' = \frac{-y}{x}$$ no issue at the bad point $x'=0$

Sanity check: $y = \pm\sqrt{R^2-x^2}$

$$y' = \pm\frac{1}{2}\frac{(-2x)}{\sqrt{R^2-x^2}} = \frac{-x}{\pm\sqrt{R^2-x^2}} = \frac{-x}{y} \checkmark$$

At $P = \left(\frac{R}{2}, \frac{\sqrt{3}}{2}R\right)$, the tangent line is $y = \frac{-1}{\sqrt{3}}\left(x - \frac{R}{2}\right) + \frac{\sqrt{3}}{2}R$

$$y' = -\frac{R}{2}\Big/\frac{\sqrt{3}}{2}R = \frac{-1}{\sqrt{3}}.$$

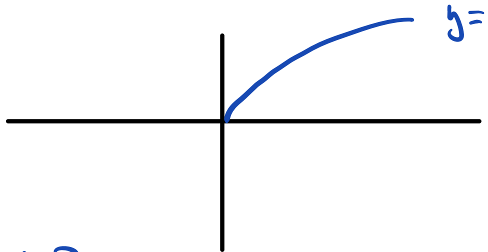
§3 Application 1: Derivative of fractional powers

INPUT: $y = x^{p/q}$ with p, q integers (coprime) & $q \neq 0$

This means $y^q = x^p$

Claim: $y' = \frac{p}{q} x^{\frac{p}{q}-1}$ (so power rule works with fractional exponents!)

Example: $\frac{p}{q} = \frac{1}{2}$



$y = \sqrt{x} = x^{1/2}$

$y' = \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2\sqrt{x}}$

Q: Why is the claim valid?

Think $y = y(x)$ & use implicit differentiation on $y(x)^q = x^p$

$q\, y^{q-1} y' = p\, x^{p-1}$ (by power rule)

$y' = \frac{p}{q} \frac{x^{p-1}}{y^{q-1}}$ as long as $y \neq 0$

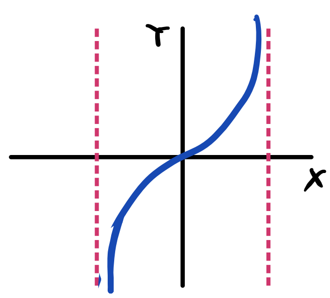
But $y^q = x^p$ so $y^{q-1} = \frac{x^p}{y} = \frac{x^p}{x^{p/q}} = x^{p-\frac{p}{q}}$

Then $y' = \frac{p}{q} \frac{x^{p-1}}{x^{p-p/q}} = \boxed{\frac{p}{q} x^{p/q-1}}$

Example: $y = \sqrt{\cos x} \rightsquigarrow y' = \frac{1}{2} \frac{1}{\sqrt{\cos x}} (\cos x)' = \frac{-\sin x}{2\sqrt{\cos x}}$.

§4. Application 2: Derivative of inverse Trig functions

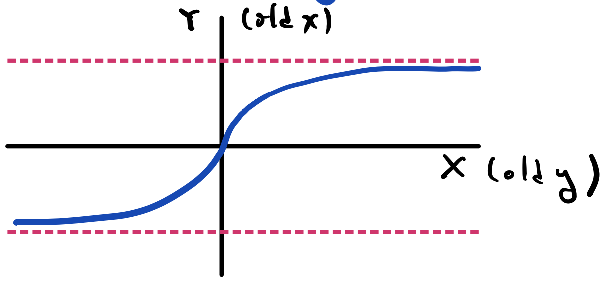
Ex: $\tan x = y$ $\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \longrightarrow \mathbb{R}$



It has an inverse function $g = g(y)$ called

$g = \arctan$, meaning $\begin{cases} g(\tan x) = x \\ \tan(g(x)) = x \end{cases}$

The graph of g is obtained by flipping the graph & the axes



$y = \arctan x$ means $\tan y = x$

GOAL: Find y' only in terms of x

Use $x = \tan(y)$ + implicit differentiation $y = y(x)$

$\frac{d}{dx} \rightsquigarrow 1 = (\tan y)' \cdot y' = \frac{1}{\cos^2 y} \cdot y'$ so $y' = \cos^2 y$

Last time

not good enough!

Can go further: $x = \tan y = \frac{\sin y}{\cos y}$

$$x^2 = \tan^2 y = \frac{\sin^2 y}{\cos^2 y} = \frac{1-\cos^2 y}{\cos^2 y} = \frac{1}{\cos^2 y} - 1$$

so $\frac{1}{\cos^2 y} = 1 + x^2$

Conclude: $\boxed{y' = \cos^2 y = \frac{1}{1+x^2}}$ for $y = \arctan(x)$