Lecture XII: §3.6 Derivatives of Higher Order
Appendix A4: The Mean Value Theorem

§1. Derivatives of Higher Order: = iterated derivatives

Simple idea: If $f: D \longrightarrow \mathbb{R}$ is differentiable, then $f': D' \longrightarrow \mathbb{R}$ is a function, defined on a (possibly smaller) set D'.

If f' is differentiable, we differentiate again, and write

$$f'' = (f')': D'' \longrightarrow \mathbb{R}, \quad x \longmapsto f''(x)$$

and so on.

Notation: y'', f'', $f^{(2)}$, $\frac{d}{dx}\left(\frac{d}{dx}f\right) = \left(\frac{d}{dx}\right)^2(f) = \frac{d^2}{dx^2}f$.

In general: $y^{(n)}$, $f^{(n)}(x)$, $\frac{d^n f}{dx^n}$ for $n \geq 1$.

Convention: $f^{(0)}(x)$ means f (no derivative!)

Example 1 Monomials (⇝ polynomials via additive rule)

- $y = c$ constant ⇝ $y' = 0$, $y'' = 0, \ldots\ldots, y^{(m)} = 0$ for all $m = 1, 2, 3, \ldots$
  (degree 0)

- $y = x^n$ $n > 0$ integer (degree n)

  $y' = n x^{n-1}$, $y'' = n(n-1)x^{n-2}$, $y^{(3)} = n(n-1)(n-2)x^{n-3}, \ldots$

Q: When does this stop? $y^{(k)} = n(n-1)\cdots(n-k+1)\,x^{n-k}$ for $k \leq n$

A: At $y^{(n+1)}$. $y^{(n+1)} = 0$ ($y^{(n)}$ is a constant).

so $y^{(m)} = 0$ for $m > n$.

Notation: We can shorten the notation if we use

$p! = p$ factorial $= p(p-1)\cdots\cdots 2\cdot 1$ for $p \geq 1$ integer

$0! = 1$ (convention).

⇝ $\boxed{y^{(k)} = \frac{n!}{(n-k)!}x^{n-k} \text{ for } k \leq n \quad \& \quad y^{(k)} = 0 \text{ for } k > n.}$

Example 2: Monomials with negative powers

$y = x^{-n} = \frac{1}{x^n}$ for $n \geq 0$ $\rightsquigarrow$ $y' = \frac{-n}{x^{n+1}}$, $y'' = \frac{-n(-(n+1))}{x^{n+2}}$,

$y^{(3)} = \frac{-n(n+1)(n+2)}{x^{n+3}}$, ... so it never ends!

In general: $y^{(k)} = (-1)^k \frac{n(n+1)\cdots n(n+k-1)}{x^{n+k}}$ for all $k>0$ integer

$$\boxed{y^{(k)} = \frac{(-1)^k}{x^{n+k}} \frac{(n+k-1)!}{(n-1)!}}$$

Example 3: Trig functions

$y = \operatorname{sen} x$, $y' = \cos x$, $y'' = -\operatorname{sen} x$, $y^{(3)} = -\cos x$, $y^{(4)} = \operatorname{sen} x$

& it repeats from here.

Similar phenomenon for $y = \cos x$.

Observe: $\operatorname{sen} x$ & $\cos x$ will both solve the differential equation $y'' = -y$.

In fact, the solutions to it are <u>all</u> of the form

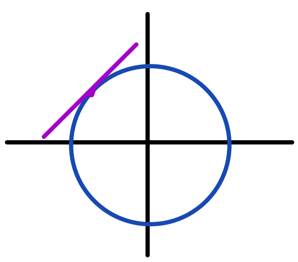
$y(x) = a \operatorname{sen} x + b \cos x$ for fixed parameters a, b

(are determined by 2 initial conditions)

$\rightsquigarrow$ We'll see this again in §9.6 (Simple Harmonic Motion).

- **Next step**: Combine higher derivatives with implicit differentiation

Ex: $x^2 + y^2 = R^2$



think $y = y(x)$

(ok, if tangent line is not vertical (see Lecture 11))

$\frac{d}{dx}$: $\boxed{2x + 2y\,y' = 0}$ (**)

$\rightsquigarrow$ $y' = \frac{-x}{y}$ if $y \neq 0$

Now: y' is differentiable, so we look at $\boxed{}$ (**) & differentiate again (implicitly!)

$$2 + 2(y'y' + yy'') = 0 \qquad \text{use } y' = -\frac{x}{y}$$

$$2 + 2\left(\left(\frac{-x}{y}\right)^2 + y\,y''\right) = 0$$

$$1 + \left(\frac{x^2}{y^2} + y \cdot y''\right) = 0$$

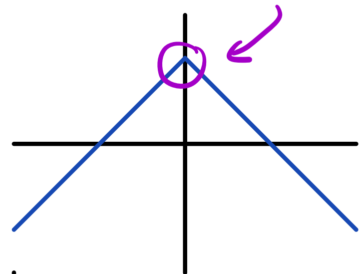
$$y\,y'' = -1 - \frac{x^2}{y^2} = \frac{-y^2 - x^2}{y^2} = \frac{-R^2}{y^2}$$

so $\boxed{y'' = \frac{-R^2}{y^2}}$ again, when $y \neq 0$

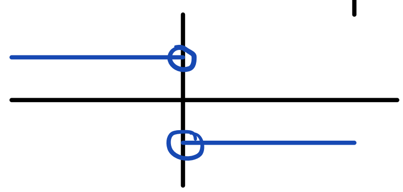
Example 4:

not differentiable at $x=0$

$$f(x) = 1 - |x| = \begin{cases} 1-x & x \geq 0 \\ 1+x & x < 0 \end{cases}$$



$$\leadsto f'(x) = \begin{cases} -1 & x > 0 \\ 1 & x < 0 \end{cases}$$

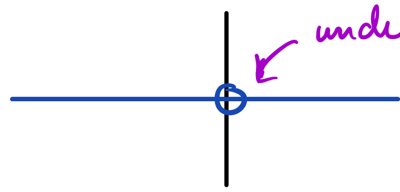


not defined at $x=0$

$$\leadsto f''(x) = 0 \quad \text{for } x \neq 0$$

$$\overset{\shortparallel}{f^{(n)}(x)} \quad \text{for all } n \geq 2.$$

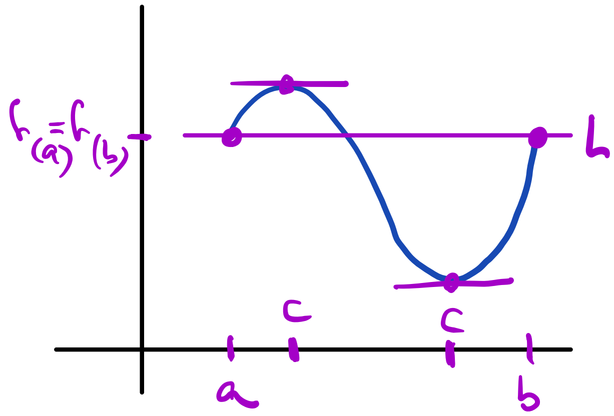
undefined at 0



§2 Appendix A4: Mean Value Theorem (MVT)

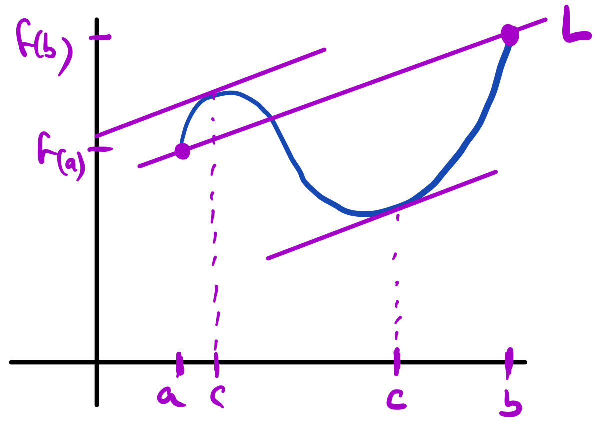
MVT: If $f:[a,b] \longrightarrow \mathbb{R}$ is continuous on $[a,b]$ & differentiable on (a,b) we can find c in (a,b) with

$$f'(c) = \frac{f(b) - f(a)}{b-a}. \quad \leadsto \text{slope of secant line } L$$



$f(a)=f(b)$

ROLLE's THM



$f(a)\neq f(b)$

We will show that we need only to check the special case when $f(a)=f(b)$. , if we take EVT for granted.

<u>ROLLE'S THEOREM</u>: If $f:[a,b]\longrightarrow\mathbb{R}$ is continuous on $[a,b]$, differentiable on (a,b) with $f(a)=f(b)$, then we can find c in (a,b) with $f'(c)=0$.

Why? Use the EVT (f continuous has maximum & minimum values on $[a,b]$)

- If f is constant, then $f'=0$ everywhere so any c will do.
- Otherwise, f is not constant, so the max & min values of f cannot agree. Since $f(a)=f(b)$ we cannot have both max & min being achieved only at $x=a$ or $x=b$, so we have some point c in (a,b) achieving one of the extreme values. Since f is differentiable, we know $f'(c)=0$.

<u>From ROLLE to MVT</u>: We need a way to view L as a horizontal line

$$L:\quad y=\frac{f(b)-f(a)}{b-a}(x-a)+f(a)$$

Equivalently: $0 = y - \left(\frac{f(b)-f(a)}{b-a}(x-a) + f(a)\right)$

We build the auxiliary function $g: [a,b] \longrightarrow \mathbb{R}$

$$g(x) = f(x) - \left(\frac{f(b)-f(a)}{b-a}(x-a) + f(a)\right)$$

($g(x)$ test membership of $(x, f(x))$ to L).

Useful properties:

- g is continuous on $[a,b]$
- g is differentiable on (a,b)
- $g(a) = g(b) = 0$

By ROLLE's THM applied to g, we have $g'(c) = 0$ for some c in (a,b)

But $g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0$ means

$$\boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}$$

General MVT: Given two functions f, g continuous on $[a,b]$ & differentiable on (a,b) with $g'(x) \neq 0$ for all x in (a,b), we can find c in (a,b) with $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

Note: $g(x) = x$ recovers MVT.

Q Why is this true?

First, we argue that $g(b) \neq g(a)$, otherwise by Rolle's Thm, we'll find c in (a,b) with $g'(c) = 0$, which contradicts our assumptions on g'.

As before, we build a new auxiliary function & apply ROLLE's THM to it to find c.

We define $h : [a,b] \longrightarrow \mathbb{R}$ as

$$h(x) = (f(b)-f(a))(g(x)-g(a)) - (f(x)-f(a))(g(b)-g(a))$$

(we build it based on the ratio we want to achieved)

Properties of h:

- h is continuous on $[a,b]$
- h is differentiable on (a,b)
- $h(a) = h(b) = 0$

By ROLLE'S THM applied to h, we have $h'(c) = 0$ for some c in (a,b)

But $h'(c) = (f(b)-f(a))\, g'(c) - f'(c)(g(b)-g(a)) = 0$

means

$$\boxed{\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}}$$