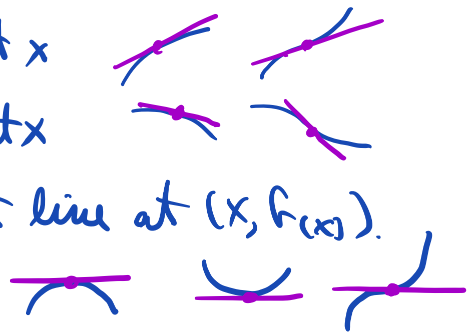
Lecture XIII: §4.1 Increasing and Decreasing Functions; max & min.

§1. Max and min Problems. Growth of functions

GOAL: Use f' & higher order derivatives to sketch the graph of f

Q: Assume f is differentiable. What does f' tell us about f?

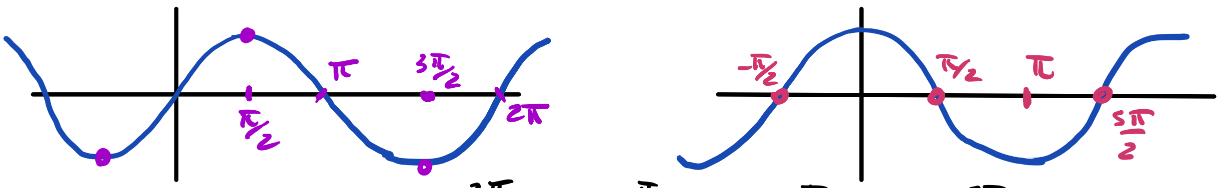
- $f'_{(x)} > 0$, then f is strictly increasing at x
- $f'_{(x)} < 0$, ——————— decreasing at x
- $f'_{(x)} = 0$, — f has a horizontal tangent line at $(x, f_{(x)})$.



Upshot: The sign of f' determines a lot!

⇝ We should look for regions where the sign of f' is constant.

EXAMPLE 1 $f_{(x)} = \sin(x)$ ⇝ $f'_{(x)} = \cos x$



		$-3\pi/2$		$-\pi/2$		$\pi/2$		$3\pi/2$	
sign of f'	+		−		+		−		+
growth of f	↗		↘		↗		↘		↗
		LOCAL MAX		LOCAL MIN		LOCAL MAX		LOCAL MIN	

(f differentiable & continuous)

EXAMPLE 2 $f_{(t)} = t^5 - 5t + 1$ ⇝ $f'_{(t)} = 5t^4 - 5 = 5(t^4 - 1)$

$= 5(t^2+1)(t^2-1)$

$= 5\underbrace{(t^2+1)}_{\text{always} > 0}(t-1)(t+1)$

Breaking points: $f'_{(t)} = 0$ ⇝ $t = \pm 1$

		-1		1	
sign of f'	+		−		+
growth of f	↗		↘		↗
		LOCAL MAX		LOCAL MIN	

⇝ get the sign by evaluating f' at convenient points (Eg $x = -2, 0, 2$)

Also $\lim_{t\to\infty} f(t) = \infty$, $\lim_{t\to\infty} f(t) = -\infty$, so f has no extreme values on $\mathbb{R}$.

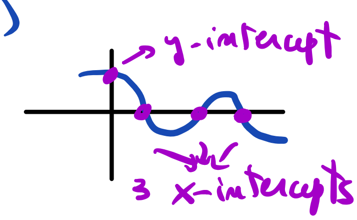
Useful Things to determine the graph of f:

① Critical points: either $f'(x) = 0$ or f is not differentiable at x (eg $x=0$ for $f(x)=|x|$)

② Critical Values: $= f(x)$ for x a critical pt.

③ Sign of $f'(x)$ between critical points & between points when f is not defined (Eg for $f(x) = \frac{1}{x(x+1)}$ ⇝ $x = 0, -1$)

④ Intercepts: $\begin{cases} x\text{-intercept} : & f(x) = 0 \\ y\text{-} \underline{\hspace{2cm}} : & f(0) \end{cases}$



⑤ $\lim_{x\to\infty} f(x)$ & $\lim_{x\to\infty} f(x)$ ⇝ potential asymptotes

⑥ Behavior of f near the points where f is not defined (Eg: $f(x) = \frac{1}{x}$ ⇝ $\lim_{x\to 0^-} f(x)$ & $\lim_{x\to 0^+} f(x)$)

⑦ Parity / Periodicity (if trig functions are involved)

Back to example 2:

$f(t) = t^5 - 5t + 1$ ⇝ t-intercepts: $t^5 - 5t + 1 = 0$ ← solve for t

They are hard to find!

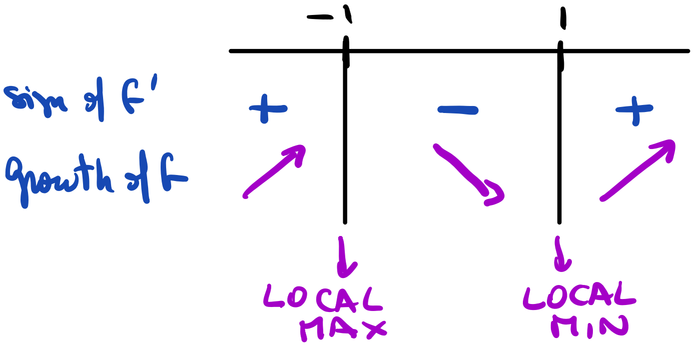
We guess the location of the t-intercepts using the Intermediate Value Theorem (IVT)

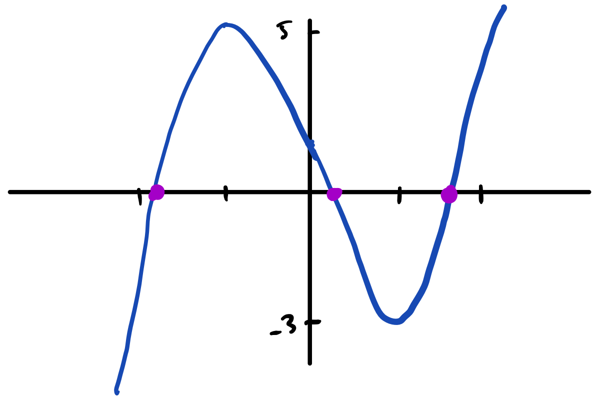
Q: How?

- $f(0) = 1 > 0$, $f(1) = -3 < 0$ & f continuous
  ⇝ so we have a zero in the interval $(0,1)$.
- $f(-1) = 5 > 0$, $f(-2) = -21 < 0$ & f continuous
  ⇝ we have another zero in the interval $(-2,-1)$.

- $f(1) = -3 < 0$, $f(2) = 23 > 0$ & f continuous
  $\leadsto$ we have another zero in the interval $(1,2)$.

We found the approximate location of 3 t-intercepts.
Growth pattern of f says we cannot have more!





<u>EXAMPLE 3</u>: $f(x) = \sec(x) = \dfrac{1}{\cos x}$ $\leadsto$ ⓪ • Periodic with period 2π

• $\dfrac{1}{\cos(-x)} = \dfrac{1}{\cos x}$, so EVEN function

• f is not defined at $x = \frac{\pi}{2}, \frac{3\pi}{2}, \dots$
$-\frac{\pi}{2}, -\frac{3\pi}{2}, \dots$

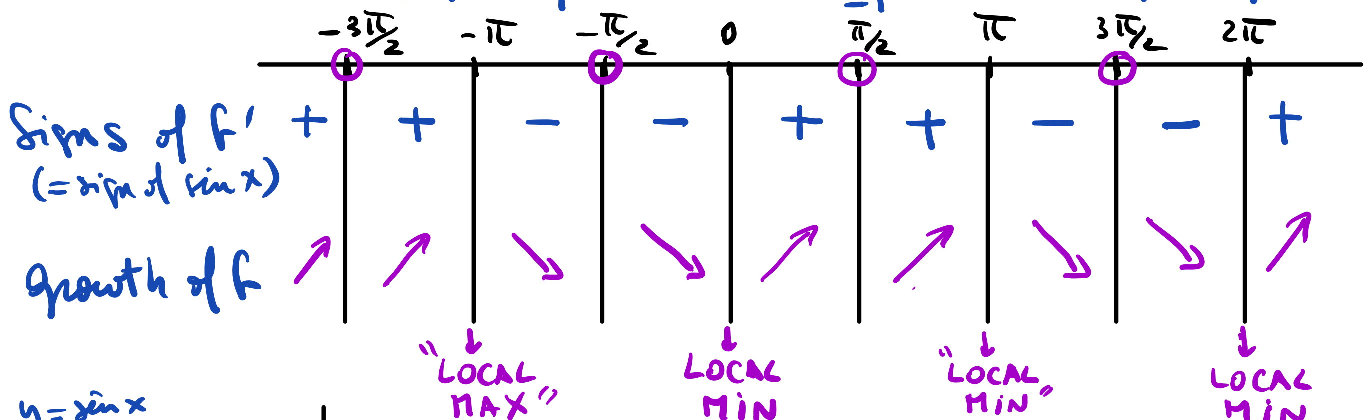
① <u>Critical Points</u>:

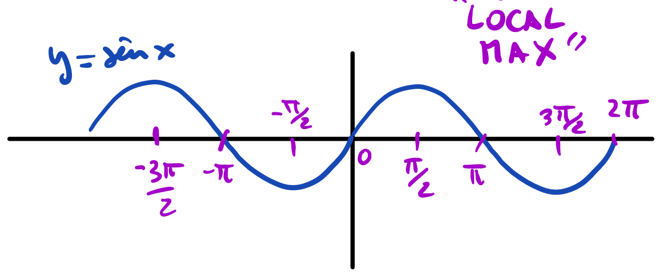
$f'(x) = -\dfrac{-\sin x}{\cos^2 x} = \dfrac{\sin x}{\cos^2 x}$ $\leadsto$ undefined at $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$
vanishes at $x = 0, \pm\pi, \pm 2\pi, \dots$

$\leadsto$ Crit. pts: $0, \pm\frac{\pi}{2}, \pm\pi, \pm\frac{3\pi}{2}, \pm 2\pi$.

② <u>Critical Values</u>: $f(0) = \frac{1}{1} = 1$, $f(\pi) = \frac{1}{-1} = -1$ & repeats periodically

③





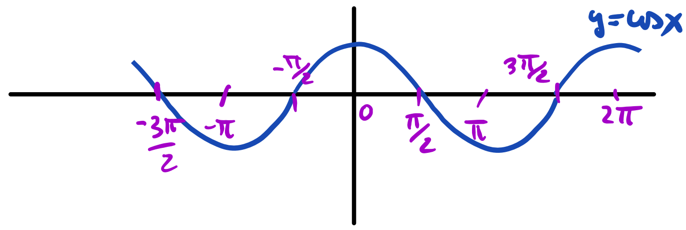
④ **Intercepts:**

- Zeros of f = none, so no x-intercepts
- y-intercept $f(0) = \frac{1}{\cos(0)} = 1$.

⑤ Behavior at ∞ & $-\infty$: We have no limits because f is periodic!

⑥ Behavior around points outside the domain of f:

$$\lim_{x \to \frac{\pi}{2}^-} \frac{1}{\cos x} = +\infty$$



$$\lim_{x \to \left(\frac{\pi}{2}\right)^+} \frac{1}{\cos x} = -\infty$$

$$\lim_{x \to \left(\frac{3\pi}{2}\right)^-} \frac{1}{\cos x} = -\infty \qquad \lim_{x \to \left(\frac{3\pi}{2}\right)^+} \frac{1}{\cos x} = +\infty$$

These give vertical asymptotes!

Graph of $\frac{1}{\cos x}$: