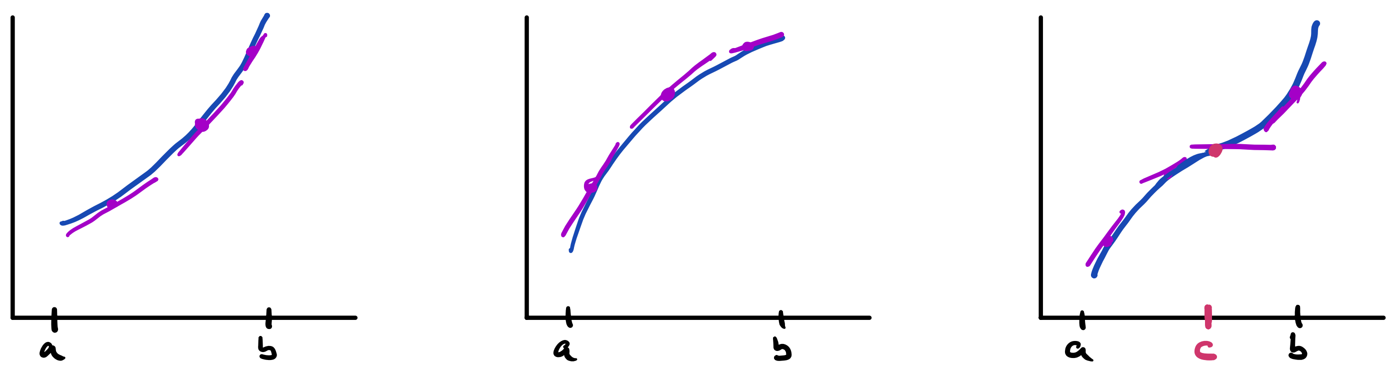
Lecture XIV: §4.2 Concavity & points of inflection

Last Time: We used f' to study the growth behavior of f, local extrema & local extreme values.

TODAY'S GOAL: Use higher order derivatives to study the convexity or "bending" of the graph of f.

Key fact: $f'' = (f')'$, so f'' gives information about the growth of f' (how are the slopes of tangent lines growing / decreasing)

3 possible scenarios:



f' str. increasing — CONCAVE UP(WARDS)

f' str. decreasing — CONCAVE DOWN(WARDS)

f' str. incr. in (a,c)
f' — decr. in (c,b)
c = inflection pt (change in convexity)

§1: Definitions & Concavity Test:

Definitions: Fix $f:[a,b] \longrightarrow \mathbb{R}$ differentiable in (a,b)

- If the graph of f lies ABOVE all of its tangent lines in (a,b) we say f is CONCAVE UP(WARDS) in (a,b)
- If the graph of f lies BELOW all of its tangent lines in (a,b) we say f is CONCAVE DOWN(WARDS) in (a,b)

Q: How to test this without drawing?

A: Use f''.

Concavity Test: Assume f' is differentiable on (a,b):

① If $f''>0$ on (a,b), then f is CONCAVE UP on (a,b)
(we write C.U.)

② If $f''<0$ on (a,b), then f is CONCAVE DOWN on (a,b)
(we write C.D.)

Q: Why does the Test work?

A (Idea) For ① f' is str. increasing & if ②, then f' is strictly decreasing. For more details, see pages 5&6.

EXAMPLE 1: $f(x) = x^3$. Find intervals where f is C.U. / C.D.

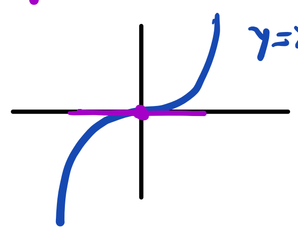
Soln: Use concavity test

$f'(x) = 3x^2 \rightsquigarrow f''(x) = 6x$ Zeroes of $f'' = 0$.

		0	
sign of f''	$-$		$+$
concavity of f	C.D.		C.U

change in concavity!

$x=0$ is an inflection point
(graph of f is on both sides of the tangent line at $(0,0)$).

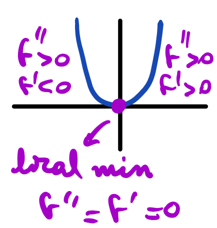


Definition: A point c in the domain of f is an inflection point if f is continuous at c & the function changes concavity at c.

Remark: Inflection points satisfy $f''(x)=0$ or f'' is NOT defined at x. In short, x must be a critical pt of f'. But we can have critical points of f' that are not inflection points. They can also be local max/min.

EXAMPLE 2: $f(x) = x^4$ has a local minimum at $x=0$.

$f'(x) = 4x^3$, $f'' = 12x^2$ so $f''(x)=0$ forces $x=0$.



	0	
sign of f''	$+$	$+$
convexity of f	CU	CU

$\rightsquigarrow x=0$ is NOT an inflection point

§2 Second Derivative Test:

Example 2 hints at the following criteria for finding local max/min.

The Second Derivative Test: Suppose $f''_{(x)}$ is continuous near c

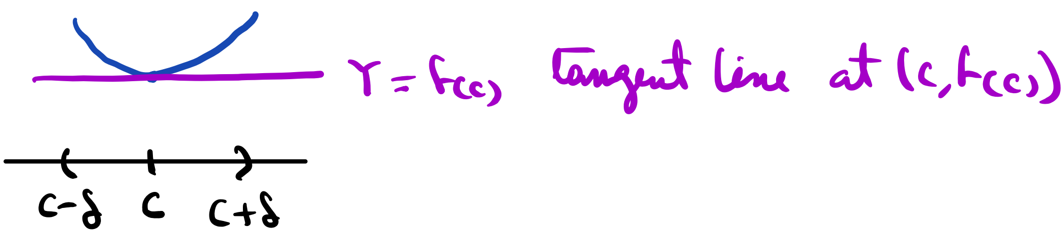
① If $f'_{(c)}=0$ & $f''_{(c)}>0$, then f has a local minimum at c

② If $f'_{(c)}=0$ & $f''_{(c)}<0$, then f has a local maximum at c

Q: Why? $f'_{(c)}=0$ says the tangent line at $(c, f_{(c)})$ is horizontal

① If $f''_{(c)}>0$, by continuity of f'' we can find a neighborhood of c where $f''>0$ on it (Fix $\varepsilon = \frac{f''_{(c)}}{2}>0$ & pick $\delta>0$ so $-\varepsilon < f''_{(x)} - f''_{(c)} < \varepsilon$, so $\underbrace{f''_{(c)}-\varepsilon}_{=\frac{f''_{(c)}}{2}>0} < f''_{(x)} < f''_{(c)}+\varepsilon$. Thus $f''_{(x)}>0$ for x in $(c-\delta, c+\delta)$)

In this interval, the graph of f sits above each tangent line (in particular, the tangent line $Y=f_{(c)}$) So $f_{(x)} > f_{(c)}$ on this interval & so c is a local minimum.



② The argument for this case is almost verbatim.

⚠ The Test says <u>NOTHING</u> when $f''_{(c)}=0$. or when $f''_{(x)}$ is not defined at c

Example: $f_{(x)}=x^3$ $x=0$ is an inflection pt, not max, not min and $f'_{(0)}=f''_{(0)}=0$.

§2 Examples:

Find the local max/min & inflection points of

(1) $f(x) = 1+3x^2-2x^3$ ⤳ differentiable up to any order!

Soln $f' = 6x-6x^2 = 6x(1-x)$, $f'' = 6-12x$

Crit points of f: $x=0$ & $x=1$

Crit points of f': $x=\frac{1}{2}$

		0		1	
sign of f'	$-$		$+$		$-$
growth of f	↘		↗		↘
		LOCAL MIN		LOCAL MAX	

		1/2	
sign of f''	$+$		$-$
convexity of f	CU		CD
		inflection pt	

Combine both tables into one

		$\boxed{0}$		1/2		$\boxed{1}$	
sign of f''	$+$		$+$		$-$		$-$
sign of f'	$-$		$+$		$+$		$-$
growth of f	↘		↗		↗		↘
convexity of f	CU		CU		CD		CD
				↳ inflection pt.			

$\square$ = crit pt of f

• Use Second Derivative Test: $f'(0)=0$ & $f''(0)>0$ ⤳ 0 is local MIN
$f'(1)=0$ & $f''(1)<0$ ⤳ 1 is local MAX

A: $x=0$ local min, $x=1$ local MAX, $x=0$ inflection point.

(2) $f(x) = \sin x$

Soln: $f'(x) = \cos x$ & $f''(x) = -\sin x$. cont.

$f''(x)=0$ for $x = 0, \pm\pi, \pm 2\pi, \ldots$

$f'(x)=0$ for $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

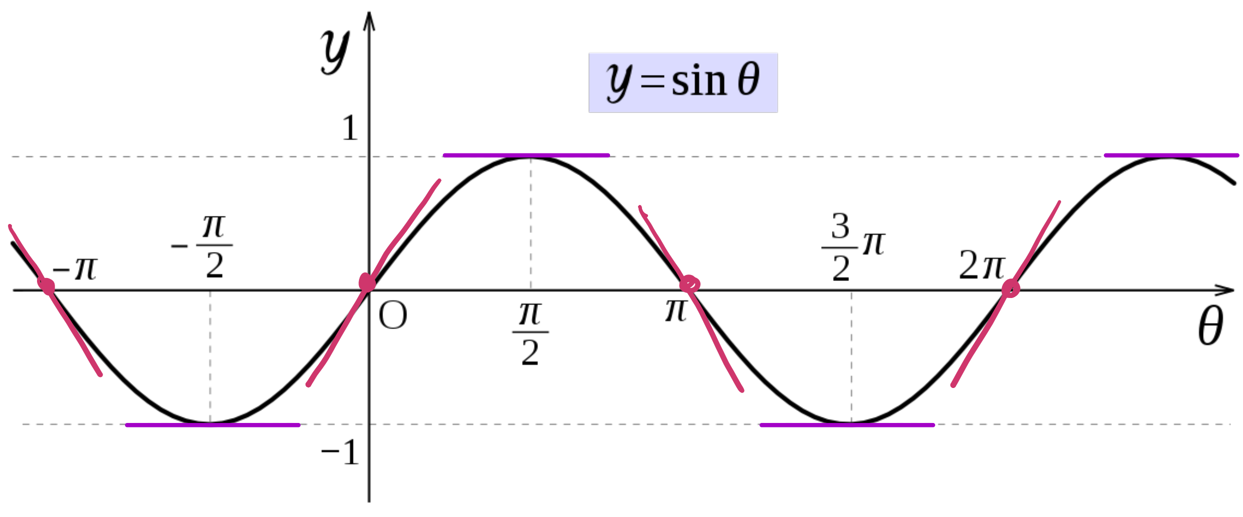
	-2π		$\boxed{-\frac{3}{2}\pi}$		$-\pi$		$\boxed{-\frac{\pi}{2}}$		0		$\boxed{\frac{\pi}{2}}$		π		$\boxed{\frac{3\pi}{2}}$		2π	
sign of f''		$-$		$-$		$+$		$+$		$-$		$-$		$+$		$+$		$-$sign sin
sign of f'		$+$		$-$		$-$		$+$		$+$		$-$		$-$		$+$		
growth of f		$\nearrow$		$\searrow$		$\searrow$		$\nearrow$		$\nearrow$		$\searrow$		$\searrow$		$\nearrow$		
convexity of f		CD		CD		CU		CU		CD		CD		CU		CU		
					$\downarrow$ inflection pt				$\downarrow$ inflection pt				$\downarrow$ inflection pt					

☐ critical points We use 2^nd derivative Test

$f''(-\frac{3}{2}\pi) < 0 \rightarrow$ local MAX ; $f''(\frac{\pi}{2}) < 0 \longrightarrow$ local MAX

$f''(-\frac{\pi}{2}) > 0 \longrightarrow$ local MIN ; $f''(\frac{3\pi}{2}) > 0 \longrightarrow$ local MIN

- Inflection Points $= 0, \pm\pi, \pm 2\pi, \ldots\ldots$



Exercise: Do the same analysis for $f(t) = t^5 - 5t + 1$ (lecture 13)

§4 Proof of the Convexity Test:

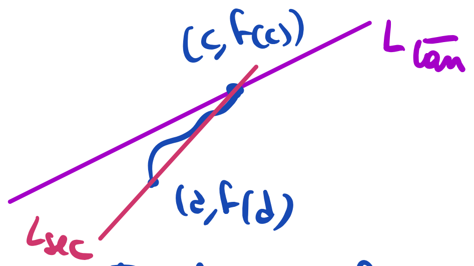
It's enough to show ① The argument for ② is very similar.

- We know that $f''(x) > 0$ on (a,b), so f' is strictly increasing $(*)$

We want to show that the graph of f sits above the tangent line at $(c, f(c))$ for any $a < c < b$

We argue by contradiction & assume this fails for some c in (a,b).
This means that we can find a point d as close to c as desired, where $(d, f(d))$ is below the tangent line L_{Tan}

CASE 1: $d<c$:



- Since f'' exists, this means f' is differentiable, so f is continuous

In particular: f is differentiable on (d,c)
f is continuous on $[d,c]$

By the Mean Value Theorem, we can find p in (d,c) with

$$f'(p) = \frac{f(c)-f(d)}{c-d} = \text{slope of the secant } L_{sec}$$

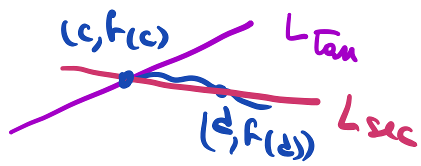
Now, $(d, f(d))$ is below L_{Tan}, $p<c$ and

$$f'(p) = \text{slope of the secant } L_{sec} > \text{slope of } L_{Tan} = f'(c)$$

So f' is <u>not</u> increasing. This contradicts our assumption (*)

CASE 2: $d>c$

We use the MVT to find p in (c,d) with $f'(p) = \dfrac{f(d)-f(c)}{d-c}$



We get $c<p$ and

$$\text{slope of } L_{Tan} = f'(c) > \text{slope of } L_{sec} = f'(c)$$

Again, this contradicts our assumption (*) that f' was increasing.

- We conclude from both cases, that no such point $(d, f(d))$ can exist, so $f(x)$ is CU near c as we wanted to show.