Lecture XV: §4.3 Applied Maximum & Minimum Problems
§4.4 More max-min problems

§1. Overview:

TODAY'S MESSAGE: The solution to many applied problems revolves around maximizing or minimizing a (constrained) function of one variable.

CHALLENGE: Translate the original problem to a concrete maximization / minimization problem

Examples: 1 In Physical Sciences, nature often wants to minimize something

(1) energy used (Hamilton's Principle of LEAST ACTION)

(2) Time required to travel from A to B (Fermat's Principle of LEAST TIME)

2 In Business, the simplest paradigm is:

(1) minimize costs.

(2) maximize profits

KEY STEPS: • Modeling = Find the equations that govern the problem

• Determine the constraints that must be met.

EASY PART: Calculus of maximizing / minimizing a continuous (or differentiable) function $f: D \longrightarrow \mathbb{R}$, where the domain D is determined by the constraints of the problem. We have 2 options for D:

1 D is bounded: $D = [a,b]$

2 D is unbounded: $[a,\infty)$, $(-\infty,b]$ or $\mathbb{R}$ (unconstrained)

Roadmap: ① Find the critical points ($f'(c)=0$ or f' is not defined at c)

② ______ values ($f(c)$ for c = critical pt)

③ Compute f at the endpoints of D, or

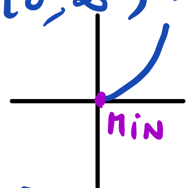
$\lim_{x\to\infty} f(x)$, respectively $\lim_{x\to-\infty} f(x)$

($D=[a,\infty)$ or $\mathbb{R}$) ($D=(-\infty,b]$ or $\mathbb{R}$)

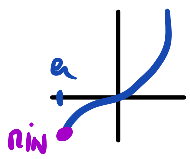
Note: Ⓐ If $D=[a,b]$ & f is continuous, then we know by EVT that f has global max & min values

Ⓑ If D is unbounded, the global max or min values may not exist (depends on what $\lim_{x\to\infty} f(x)$ and/or $\lim_{x\to-\infty} f(x)$ are)

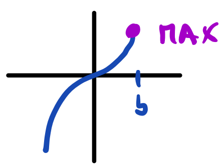
Examples ① $f(x) = x^2$ $f:[0,\infty)\to\mathbb{R}$ has global min, but no global max



② $f(x) = x^3$: $f:[a,\infty)\to\mathbb{R}$ has global min, but no global max



$f:(-\infty,b]\to\mathbb{R}$ has global max, but no global min



③ $f(x) = \sin(x)$ has both global minimum & maximum values on $\mathbb{R}$

Strategy for Modeling:

① Understand / Interpret the word problem.

② Make a careful sketch if the problem is geometric. Don't include any unwanted assumptions (eg symmetries) that aren't granted

③ Label the figure with the given data & suggestive names for the variables.
④ Write down the equations involved and what needs to be maximize / minimize. Then, transform the equation into a single variable one.
⑤ Solve the max/min problem & make sure the answer makes sense in the context of the word problem

§2. Examples:

EXAMPLE 1: Show that the rectangle with maximum area for a fixed perimeter is a square.

Solution:

l

h $\qquad$ h

l

(1) $\underset{(l,h)}{\text{Area}} = l \cdot h$ $\rightsquigarrow$ To be maximized

(2) Perimeter $= 2l + 2h = 2(l+h) = P$

$P \geq 0$ is fixed.

Constraints:
- $l, h \geq 0$
- $l + h = \frac{P}{2}$ fixed

Use (2) to turn Area(l,h) into a single variable expression

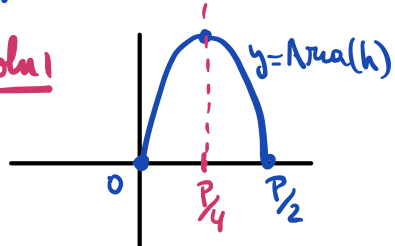
Q: How? A: Solve for l $\boxed{l = \frac{P}{2} - h}$ & replace this value in Area.

$$\text{Area} = \text{Area}(h) = \left(\frac{P}{2} - h\right) h = \frac{P}{2} h - h^2$$

Constraints : $h \geq 0$ & $\frac{P}{2} - h \geq 0$ so $\boxed{0 \leq h \leq \frac{P}{2}}$

$\rightsquigarrow$ Area : $D = [0, \frac{P}{2}] \longrightarrow \mathbb{R}$ is a continuous & differentiable function

Soln 1



graph says max is achieved at $h = \frac{P}{4}$ so $l = \frac{P}{2} - h = \frac{P}{4}$.

Solnz Maximize $A_{(h)} = \text{Area}(h) = \frac{P}{2}h - h^2$ over $0 \leq h \leq \frac{P}{2}$

We have a global maximum by the EVT.

1. Critical Points: $A'(h) = \frac{P}{2} - 2h = 0$ so $h = \frac{P}{4}$
2. Critical Values: $A\left(\frac{P}{4}\right) = \frac{P}{2}\frac{P}{4} - \left(\frac{P}{4}\right)^2 = \frac{P^2}{8} - \frac{P^2}{16} = \frac{P^2}{16}$
3. Values at endpoints: $A(0) = 0 = A\left(\frac{P}{2}\right)$
4. Comparison: $A\left(\frac{P}{4}\right)$ is the winner, so $h = \frac{P}{4}$ & $l = \frac{P}{2} - \frac{P}{4} = \frac{P}{4}$

Conclude $l = h = \frac{P}{4}$ so the rectangle is a square. $\square$

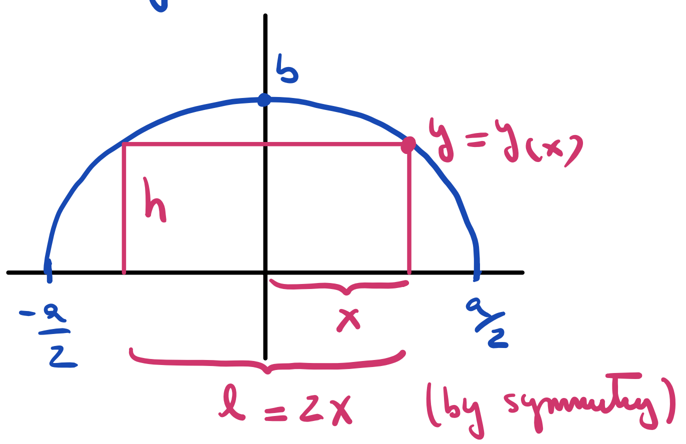
EXAMPLE 2: Given 2 positive constraints a & b, consider the region between the parabola $a^2y = a^2b - 4bx^2$ and the x-axis. Find the base and height of the largest rectangle (in area) with lower base on the x-axis & upper vertices on the parabola.

Solution We draw the parabola by computing the x-intercepts

$a^2y = a^2b - 4bx^2$ & $y = 0$, so $4bx^2 = a^2b$
$x = \pm\frac{a}{2}$

Symmetry then says the vertex of the parabola is the y-intercept

$a^2y = a^2b - 4bx^2$ & $x = 0$, so $a^2y = a^2b$
$y = b$.



$y = \frac{a^2b - 4bx^2}{a^2}$, so

$h = b - \frac{4bx^2}{a^2}$

Area $= lh = 2xh$

$A_{(x)} = 2x\left(b - \frac{4bx^2}{a^2}\right)$

Constraints: $0 \leq x \leq \frac{a}{2}$

- We have a bounded problem: maximize $A(x)$ over $[0, \frac{a}{2}]$

1. Critical Pts: $A'(x) = 2b - \frac{8b3x^2}{a^2} = 2b\left(1 - \frac{12}{a^2}x^2\right) = 0$

   Since $b \neq 0$, we get $x = \pm \frac{a}{\sqrt{12}}$.

   Since $x \geq 0$, we discard the negative solution

2. Critical Values: $A\left(\frac{a}{\sqrt{12}}\right) = \frac{2a}{\sqrt{12}}\left(b - \frac{4 \cdot b}{a^2}\frac{a^2}{12}\right) = \frac{4ba}{3\sqrt{12}}$

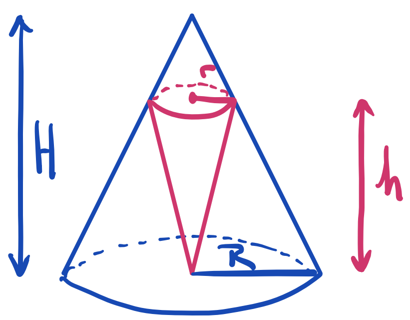
3. Endpoint Values: $A(0) = 0 = A\left(\frac{a}{2}\right)$

4. Comparison: $\frac{a}{\sqrt{12}}$ wins, so base $= \frac{2a}{\sqrt{12}} = \boxed{\frac{a}{\sqrt{3}}}$

   height $= b - \frac{4b}{a^2}\frac{a^2}{12} = \boxed{\frac{2b}{3}}$

EXAMPLE 3: A cone with height h is inscribed in a larger cone with height H so that its vertex is at the center of the base of the larger cone. Show that the inner cone has maximum volume when $h = H/3$.

Soln:



- Big cone: height $H \geq 0$, radius $R \geq 0$ } fixed
- Small cone: height $h \geq 0$, radius $r \geq 0$ } variables

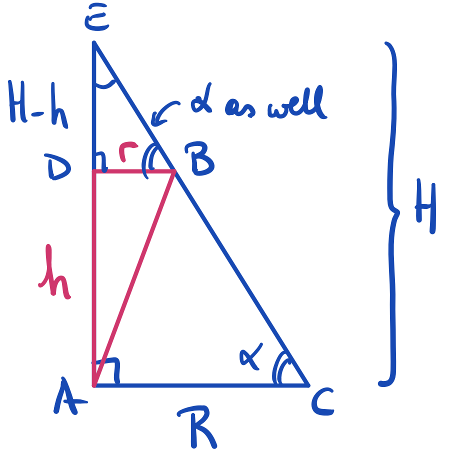
inscribed = inside + touching the boundary

Constraints: $0 \leq h \leq H$ & $0 \leq r \leq R$

Vol Cone $(r, h) = \frac{\pi r^2 h}{3}$ we want to maximize it

Q How to find a relation between r & h?

A: Use geometry of triangular vertical cross section.



DB // AC because small cone is inscribed in the larger cone.

So $\tan\alpha = \dfrac{H-h}{r} = \dfrac{H}{R}$

This gives $\boxed{r = \dfrac{R}{H}(H-h)}$

Replace this expression in Vol Cone to get a one-variable function

$$Vol(h) = \frac{\pi}{3}\underbrace{\frac{R^2(H-h)^2}{H^2}}_{=r^2} h$$

cubic polynomial in h

- Vol is continuous and differentiable
- Constraints : $0 \le h \le H$ ($0 \le r \le R$ translate to $0 \le \frac{R}{H}(H-h) \le R$, so $0 \le H-h \le H$, $0 \le h \le H$)

Now we find the global max, which exists by EVT:

- $$Vol'(h) = \frac{\pi R^2}{3H^2}\left(2(H-h)(-1)h + (H-h)^2\right)$$
$$= \frac{\pi R^2}{3H^2}(H-h)(-2h+H-h)$$
$$= \frac{\pi R^2}{3H^2}(H-h)(H-3h)$$

So $Vol'(h) = 0$ for $h = H$ or $h = \frac{H}{3}$

- $Vol(H) = 0$ & $Vol\left(\frac{H}{3}\right) = \frac{\pi}{3}\frac{R^2}{H^2}\left(\frac{2}{3}H\right)^2\frac{H}{3} = \frac{4}{81}\pi R H > 0$
- End points $Vol(0) = Vol(H) = 0$
- Compare : The winner is $h = \frac{H}{3}$.