Lecture XVI: §4.5 Related Rates

§1. Overview:

Idea: • We have 2 or 3 quantities linked together by a constraint (typically, an equation)

• The system is changing ("in motion") and so all quantities are changing as well ("with time").

GOAL: Compute the rate of change of one quantity in terms of the known rate of change of the rest.

TOOLS: ① Implicit differentiation

② Chain Rule + Substitution of known values.

• Only difficulty: Modeling the problem

• After modeling, we follow the same strategy as with max/min problems

① Draw diagrams, label figures. Identify data & units of measure

② NOVELTY: Find the relationship between the varying quantities

(Usually, there is nothing to maximize / minimize)

⚠ Don't fix quantities that are changing until the very end.

§2. Examples:

EXAMPLE 1: Air is being pumped into a spherical balloon whose volume increases at a constant rate of $8\,\frac{ft^3}{min}$. Find the rate of growth of the radius of the balloon when (a) its radius is 4 ft; (b) its volume is 1 ft^3

Solution: $Vol(r) = \frac{4}{3}\pi r^3$ Here: $r = r(t)$ changes with time

Know: $8 = \frac{dV}{dt} = \frac{4}{3}\pi\, 3r^2 \boxed{r'}$ & want to solve for r'

So $r' = \frac{dr}{dt} = \frac{8}{4\pi}\frac{1}{r^2} = \frac{2}{\pi r^2}$

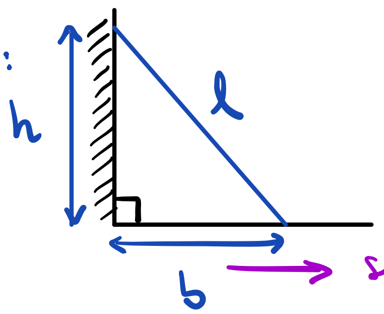
(a) When $r = 4$ ft, $r' = \frac{dr}{dt} = \frac{2}{\pi 4^2} = \frac{1}{8\pi}$ ft/min

(b) When Vol $= \frac{4}{3}\pi r^3 = 1$ ft^3, the $r = \sqrt[3]{\frac{3}{4\pi}}$ ft

Conclude: At this point in time: $r' = \frac{2}{\pi}\left(\frac{4\pi}{3}\right)^{2/3} \frac{\text{ft}}{\text{min}}$

EXAMPLE 2: Find the speed at which a ladder slides down a wall if it slides away from the wall at a fixed rate.

Solution:



l = length of the ladder (fixed)

$h = h(t)$ = distance to the floor

$b = b(t)$ = ——— wall

Equation relating the 3 quantities: $\boxed{l^2 = h(t)^2 + b(t)^2}$ (*)

(Pythagoras Theorem)

Know: rate at which the ladder slides AWAY from the WALL $= b'(t)$.

GOAL: Compute $h'(t)$.

Use implicit differentiation in (*) to get $h'(t)$

$$0 = 2h(t)\,h'(t) + 2b(t)\,b'(t) \quad \text{\& solve for } h'$$

$$\boxed{h'(t) = -\frac{b(t)\,b'(t)}{h(t)}}$$

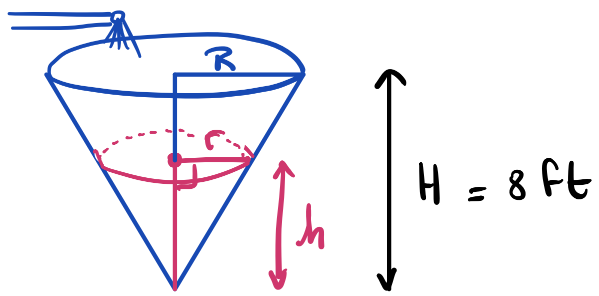
Example If $l = 13$ ft, $b'(t) = 6 \frac{\text{ft}}{\text{min}}$ & $b(t) = 5$ ft

Then $h(t) = \sqrt{l^2 - b^2(t)} = \sqrt{13^2 - 5^2} = 12$ ft & so

$$h'(t) = -\frac{5 \cdot 6 \text{ ft}^2/\text{min}}{12 \text{ ft}} = -\frac{5}{2} \text{ ft/min}$$

EXAMPLE 3: A conical water tank with its vertex down is 8 ft high & 4 ft in diameter at the top. The tank is full, and water leaks through a hole in the bottom at a rate of $1\ \mathrm{ft}^3/\mathrm{min}$. Find the rate at which the water level is falling when the tank is $\frac{7}{8}$ empty. Level = measured by height or volume

Solution:



Constraints

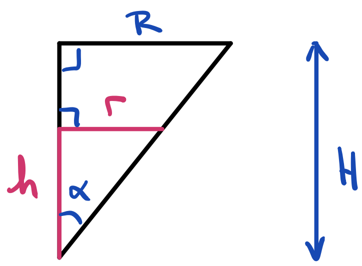
- Tank: $\frac{7}{8}$ empty $= \frac{1}{8}$ full

$R = \frac{4\ \mathrm{ft}}{2} = 2\ \mathrm{ft}$

$H = 8\ \mathrm{ft}$

Full : $h = H$, empty : $h = 0$,

The triangular cross section reveals the relationship between h & r



$$\frac{R}{H} = \frac{r}{h} = \tan\alpha \quad \text{so} \quad \frac{r}{h} = \frac{2}{8} = \frac{1}{4}$$

$$\boxed{h = 4r}$$

Relation $h = 4r$ & $\mathrm{Volume}(t) = \dfrac{\pi r^2 h}{3} = \dfrac{4\pi r^3(t)}{3} = \dfrac{\pi h^3(t)}{3 \cdot 16}$

Q: What are the constraints on the time?

- Option 1: $\frac{1}{8}$ Full : $h = \frac{1}{8} H = 1\ \mathrm{ft}$ (Level measured by height)

Conclusion 1: We need to focus on the time t where $h(t) = 1$.

⇝ Think of Volume as a function of $h(t)$: $\mathrm{Vol}(t) = \dfrac{\pi\, h^3(t)}{3 \cdot 16}$

The Chain Rule gives $\underbrace{\mathrm{Vol}'(t)}_{= -1\ \mathrm{ft}^3/\mathrm{min}} = \dfrac{3\pi\, h^2(t)\, h'(t)}{3 \cdot 16} \overset{h(t)=1 \text{ at the relevant time}}{=} \dfrac{\pi h'}{16}\ \mathrm{ft}^2$

so $\boxed{h' = \dfrac{-16}{\pi}\ \mathrm{ft}/\mathrm{min}}$

- Option 2: $\frac{1}{8}$ Full means Vol water $= \frac{1}{8}$ Vol when full (Volume level)

$$Vol = \frac{\pi r^2 h}{3} \text{ (water) vs Full vol} = \frac{\pi R^2 H}{3} = \frac{\pi\, 4 \cdot 8}{3} = \frac{32\pi}{3}$$

So $Vol = \frac{1}{8}$ Full Vol means $\frac{\pi r^2 h}{3} = \frac{32\pi}{3}$, so

$$h = \frac{32\pi}{3r^2}.$$

Use $h = 4r$ to get $h = \frac{32\pi}{3\left(\frac{h}{4}\right)^2} = \frac{32\pi}{\frac{3h^2}{16}}$

$$h^3 = \frac{32 \cdot 16}{3}\pi \qquad h = \sqrt[3]{\frac{32 \cdot 16}{3}\pi} \text{ ft}$$

Conclusion 2. We must focus on t giving $h(t) = 64\sqrt{\frac{\pi}{3}}$ ft

Now we use the Chain Rule on $Vol(t) = \frac{\pi\, h^3(t)}{3 \cdot 16}$

$$\underset{\substack{\shortparallel \\ -1\frac{ft^3}{min}}}{Vol'(t)} = \frac{3\pi\, h^2(t) h'(t)}{3 \cdot 16} = \frac{\pi}{16}\, 64^2 \left(\frac{\pi}{3}\right)^{2/3} h'(t)$$

so $\boxed{h'(t) = -\frac{1}{256}\sqrt[3]{\frac{9}{\pi}}\ \text{ft/min}}$

EXAMPLE 4: Same data but want the rate at which the height level is falling when we've collected 36π ft^3 of water.

Solution: Now we have $\underset{-1\frac{ft^3}{min}}{\overset{\shortparallel}{Vol'}} = \frac{\pi h^2}{16} h'$ & we know $Vol = \frac{\pi h^3(t)}{48} = 36\pi$ so $h^3(t) = 36 \cdot 48 = 12^3$

$h(t) = 12$

Then $h' = \frac{-16}{\pi h^2} = \frac{-16}{\pi (12)^2} = \boxed{\frac{-1}{9\pi}\ \frac{ft}{min}}.$