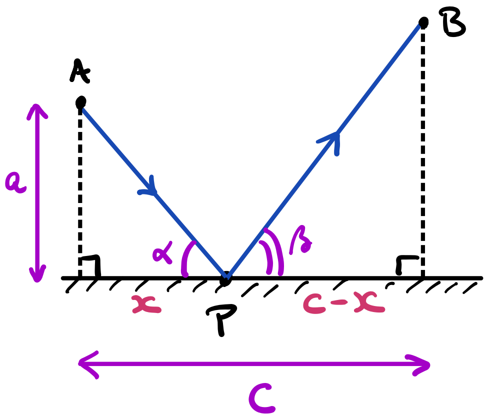
Lecture XVII: §4.4 Reflection & refraction

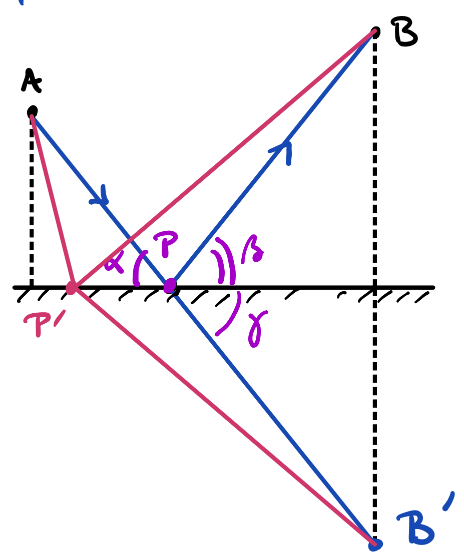
§1. Law of Reflection:



A ray of light travels from A to P, bounces on the mirror surface to reach B in the shortest amount of time. Then,

Proposition: $\alpha = \beta$

Proof 1: Set B' = mirror image of B & draw P as the intersection of the line AB' with the mirror surface.



Then $\alpha = \gamma$ (opposite angles) & $\gamma = \beta$

because $\mathrm{dist}(B, \text{mirror}) = \mathrm{dist}(B', \text{mirror})$

From this we get $\alpha = \gamma = \beta$.

Q: Why is this the shortest path?

A: We know the speed of light is constant so the fastest way is also the shortest way.

- We compute the total length:

$$\ell(APB) = AP + PB = AP + PB' = AB'.$$

- Now, for any other point P' on the mirror, we can draw the same 3 lines AP', $P'B$ & $P'B$.

$$\ell(AP'B) = AP' + P'B = AP' + P'B'$$

But $AP' + P'B > AP + PB'$ unless $P' = P$.

(shortest path between 2 points is a straight line!)

Conclusion: The P we pick minimizes the distance travelled, so $\alpha = \beta$ must be true.

Proof 2: View this as a minimization problem & use Calculus!

The length of the path depends on the location of P. We set x to be the distance from P to the left & write $P = P(x)$. Then:

$$L(x) = AP(x) + P(x)B = \sqrt{x^2+a^2} + \sqrt{(c-x)^2+b^2}.$$

Constraint: $0 \leq x \leq c$.

Note: $a, b > 0$ so $L(x)$ is continuous & differentiable.

By EVT it has a minimum $\begin{cases} \text{endpoint} \\ \text{crit pt} \end{cases}$

(1) Endpoints $L(0) = a + \sqrt{b^2+c^2}$ & $L(c) = \sqrt{c^2+a^2} + \sqrt{b^2}$.

$$L'(x) = \frac{2x}{2\sqrt{x^2+a^2}} + \frac{1}{2}\frac{-2(c-x)}{\sqrt{(c-x)^2+b^2}} = \frac{x}{\sqrt{a^2+x^2}} - \frac{(c-x)}{\sqrt{(c-x)^2+b^2}}$$

NOTE: $L'(0) = \frac{-c}{\sqrt{c^2+b^2}} < 0$ & $L'(c) = \frac{c}{\sqrt{a^2+c^2}} > 0$,

so L decreases near 0 & increases near c. This ensures that endpoints cannot be minimizing points!

(2) Critical Points: $L'(x) = 0$ gives $\frac{x}{\sqrt{a^2+x^2}} = \frac{c-x}{\sqrt{(c-x)^2+b^2}}$.

We square both sides: $\frac{x^2}{a^2+x^2} = \frac{(c-x)^2}{(c-x)^2+b^2}$

We invert: $\frac{a^2+x^2}{x^2} = \frac{(c-x)^2+b^2}{(c-x)^2}$

$$\cancel{1} + \left(\frac{a}{x}\right)^2 = \cancel{1} + \left(\frac{b}{c-x}\right)^2$$

Invert again: $\left(\frac{x}{a}\right)^2 = \left(\frac{c-x}{b}\right)^2$ $\rightsquigarrow$ quadratic in x we can solve!

But we don't need to solve for x. Remember: we want to confirm that $\alpha = \beta$ (for this minimizing value for x).

Here: $\tan\alpha = \frac{a}{x}$ & $\tan\beta = \frac{b}{c-x}$, with $(\tan\alpha)^2 = (\tan\beta)^2$

& $\tan\alpha, \tan\beta > 0$ because $a, b, x, c-x > 0$ & $\alpha, \beta \in [0, \frac{\pi}{2})$.

We conclude: $\alpha = \beta$ at a critical point of L.

NOTE: At a critical point we get a minimum by the 2nd Derivative Test:

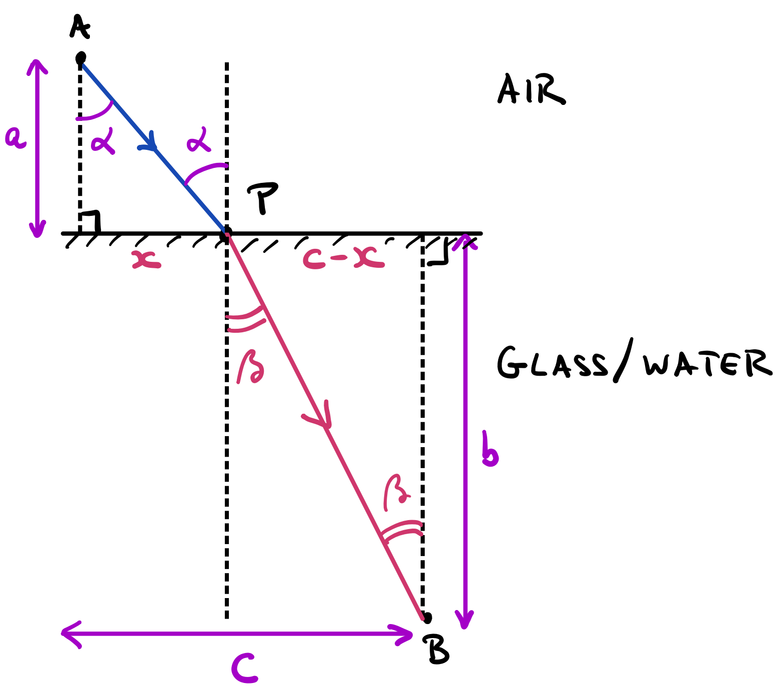
$$L' = \frac{1}{\sqrt{\left(\frac{a}{x}\right)^2 + 1}} - \frac{1}{\sqrt{1 + \left(\frac{b}{c-x}\right)^2}}$$

$$\rightsquigarrow L'' = \frac{1}{2}\frac{2\,\frac{a}{x}\cdot\frac{(-1)}{x^2}}{\left(\left(\frac{a}{x}\right)^2+1\right)^{3/2}} - \frac{1}{2}\frac{2\,\frac{b}{(c-x)^3}(-1)}{\left(1+\frac{b}{c-x}\right)^{3/2}} = \frac{a^2}{(a^2+x^2)^{3/2}} + \frac{b^2}{((c-x)^2+b^2)^{3/2}} > 0$$

§2 Law of Refraction:

Q: What happens when a ray of light travels through 2 different media (air & water / glass, ...)?

A: Again, we want to minimize the time it takes to travel from A (in one medium) to B (in a different medium).



Write v_a = speed of light in air.

v_b = ______ in glass/water

We assume $v_a, v_b > 0$.

x = distance from P to the left

Constraint: $0 \leq x \leq c$

Write α & β for the 2 relevant angles (in the picture). Note that they come in pairs. & they depend on the position of P, so on the value x.

Snell's Law: $\dfrac{\sin\alpha}{\sin\beta} = \text{constant} = \dfrac{v_a}{v_b}$ (called the index of refraction)

This constant only depends on v_a & v_b, not on the position of A or B.

NOTE: If $v_a = v_b$ we recover the Law of Reflection.

Proof As usual, we set this as a problem of minimizing the time it takes to travel the distance from A to B.

- Time in the air: $T_{air}(x) = \dfrac{\text{distance}}{|\text{velocity}|} = \dfrac{AP}{v_a} = \dfrac{\sqrt{a^2+x^2}}{v_a}$
- Time in the glass/water $T_{g/w}(x) = \dfrac{\text{distance}}{|\text{velocity}|} = \dfrac{PB}{v_b} = \dfrac{\sqrt{b^2+(c-x)^2}}{v_b}$

So: Total Time $T(x) = T_{air}(x) + T_{g/w}(x)$

$$\boxed{T(x) = \frac{(a^2+x^2)^{1/2}}{v_a} + \frac{(b^2+(c-x)^2)^{1/2}}{v_b}}$$

We want to minimize $T(x)$ subject to $0 \leq x \leq c$.

As before: T is continuous & differentiable on $[0,c]$

By the EVT we have a minimum.

(1) Endpoints: $T(0) = \dfrac{a}{v_a} + \dfrac{(b^2+c^2)^{1/2}}{v_b}$; $T(c) = \dfrac{(a^2+c^2)^{1/2}}{v_a} + \dfrac{b}{v_b}$

$$T'(x) = \frac{x}{v_a(a^2+x^2)^{1/2}} - \frac{(c-x)}{v_b(b^2+(c-x)^2)^{1/2}}$$

$T'(0) = \dfrac{-c}{v_b(b^2+c^2)^{1/2}} < 0$ & $T'(c) = \dfrac{c}{v_a(a^2+c^2)^{1/2}} > 0$

so T is str. decreasing at 0 & st. increasing at c. Thus, the minimum cannot be achieved at an endpoint. It must be a critical point!

(2) Critical point: $T'(x) = 0$ gives $\dfrac{1}{v_a}\underbrace{\boxed{\dfrac{x}{(a^2+x^2)^{1/2}}}}_{=\sin\alpha(x)} = \dfrac{1}{v_b}\underbrace{\boxed{\dfrac{c-x}{(b^2+(c-x)^2)^{1/2}}}}_{=\sin\beta(x)}$

Conclusion: At the minimum, we have $\dfrac{\sin\alpha(x)}{\sin\beta(x)} = \dfrac{v_a}{v_b} = \text{constant}$.

Note: Use 2nd Derivative Test to confirm we get a minimum since
$$T''(x) = \frac{1}{v_a}\frac{a^2}{(a^2+x^2)^{3/2}} + \frac{1}{v_b}\frac{b^2}{(b^2+(c-x)^2)^{3/2}} > 0.$$