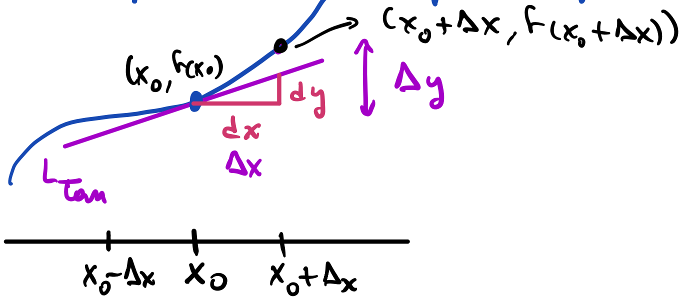
Lecture XVIII: §5.2 Differentials & tangent line approximations

§1. Linear approximations & differentials:

IDEA: Near a given point, the tangent line to a graph at this point is a good approximation of this graph.



Recall: $\Delta y = f(x_0+\Delta x) - f(x_0)$

Δx = increment in x-variable

- Increase in the graph $= (\Delta x, \Delta y)$
- ——— tangent $= (dx, dy)$

$$L_{tan}: \quad y = f'(x_0)(x-x_0) + f(x_0)$$

$$\boxed{\begin{aligned} dy &= y - f(x_0)\Big|_{x=x_0+\Delta x} = f'(x_0)\,\Delta x = f'(x_0)\,dx \\ dx &= x - x_0\Big|_{x=x_0+\Delta x} = \Delta x \end{aligned}}$$ differentials

- The notation dx, dy is inspired by Leibniz:

$$y = f(x) \qquad y' = f'(y) = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$ "as if" $\lim\limits_{\Delta x\to 0} \Delta y = dy$ & $\lim\limits_{\Delta x \to 0} \Delta x = dx$

(Of course, both limits are 0 !)

- We want to give a meaning to $f'(x) = \frac{dy}{dx}$. This is precisely what differentials do! Q: How?

A: The differential $\boxed{dx}$ is an independent variable, and $\boxed{dy}$ is defined in terms of dx by $\boxed{dy := f'(x_0)\,dx}$

Summary: • $\Delta f = \Delta y$ = change along the graph $y=f(x)$ (a curve!) as x changes to $x+\Delta x$

• $df = dy$ = ——— tangent line as x changes to $x + dx$ $(= x+\Delta x)$

• **Remark** If f is linear, $y = mx + b$ for fixed parameters m & b

Then $\left.\begin{array}{l} \Delta y = m\,\Delta x \quad , \; y' = m \\ \Delta x = dx \end{array}\right\}$ so $dy = y'dx = m\,dx = m\,\Delta x = \Delta y$
& $\frac{\Delta y}{\Delta x} = m = \frac{dy}{dx}$.

• Idea behind linear approximations: pretend that f is linear & use the tangent line as an "approximation" of f.

The catch will be to see what the error in the approximation, but we'll deal with this when we discuss Remainder formulas for Taylor series. For today, we take for granted that the error is small.

Examples: • $f(x) = x^2 \rightsquigarrow f'(x) = 2x$ so $df = dy = 2x\,dx$
$\Delta f = \Delta y = (x+\Delta x)^2 - (\Delta x)^2$
$= x^2 + 2x\,\Delta x$

• $f(x) = \sin x \rightsquigarrow f'(x) = \cos x$ so $df = dy = \cos x\,dx$
$\Delta f = \Delta y = \sin(x+\Delta x) - \sin x$

§2 Differentiation Formulas in differential notation:

Fix u, v function of 1 variable x.

1. Power Rule: $y = u^n \rightsquigarrow dy = n u^{n-1} du$

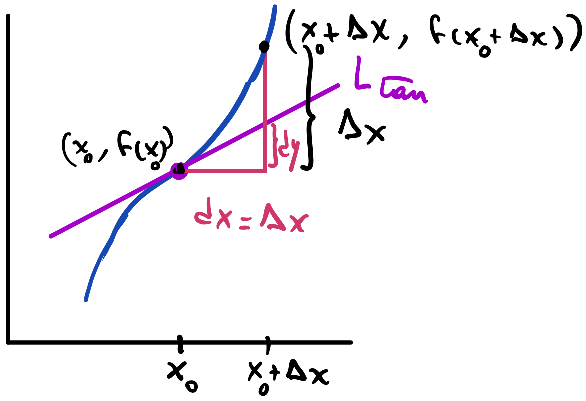
2. Product Rule: $y = uv \rightsquigarrow dy = d(uv) = u\,dv + v\,du$
[Q: Why? A: $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ & multiply both sides by dx]

3. Quotient Rule: $y = \frac{u}{v} \rightsquigarrow dy = d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$

4. Chain Rule: $y = f(u)$ $u = g(x)$ $\rightsquigarrow dy = f'(u)\,du$
$du = g'(x)\,dx$

so $dy = f'(u)\,g'(x)\,dx = f'(g(x))\,dx$.

§3. Tangent line approximation:



Near the tangency point $(x_0, f(x_0))$, the graph of f is very close to the tangent line. As $\Delta x = dx$ becomes closer to 0, then the

secant line (Δy data) $\longrightarrow$ tangent line (dy data)

Conclusion: For SMALL values of $\Delta x = dx$, the change in the tangent line is a good approximation to the change in $f(x)$.

$$\Delta y = f(x_0+\Delta x) - f(x_0) \simeq dy$$

Equivalently: $$\boxed{f(x_0+\Delta x) \simeq f(x_0) + dy = f(x_0) + \underbrace{f'(x_0)\,dx}_{\text{LINEAR in } dx}}$$

Definition: The linear function $L(x) = f(x_0) + f'(x_0)(x - x_0)$ is called the linear approximation of f at x_0.

- We use $L(x)$ to approximate $f(x)$ for x near x_0.
- Applications: ① Approximate values of square roots, cube roots, etc.
  ② Compute impact of measurement errors

EXAMPLE 1: Find the linear approximation of $f(x) = \sqrt{1-x}$ at $a=0$ & use it to estimate $\sqrt{0.9}$ & $\sqrt{0.99}$.

Solution: $f'(x) = \dfrac{1}{2\sqrt{1-x}}(-1) \leadsto f'(0) = \dfrac{-1}{2}$, $f(0) = \sqrt{1-0} = 1$

$$L(x) = f(0) + f'(0)(x-0) = 1 - \tfrac{1}{2}x$$

So $f(0.1) = \sqrt{0.9} \approx L(0.1) = 1 - \frac{1}{2}0.1 = 1 - 0.05 = 0.95$

$f(0.01) = \sqrt{0.99} \approx L(0.01) = 1 - \frac{1}{2}0.01 = 1 - 0.005 = 0.995$.

EXAMPLE 2: Find the linear approximation of $\sin(x)$ at $x=0$

Solution: $f'(x) = \cos x \rightsquigarrow f'(0) = \cos 0 = 1$, $f(0) = \operatorname{sen}(0) = 0$

$$L(x) = f(0) + f'(0)(x-0) = 0 + 1\cdot x = x.$$

Observe: This says $L(x) = x$ is a good approximation of $\operatorname{sen} x$ near 0. This is consistent with the fact that $\lim_{x\to 0} \frac{\operatorname{sen} x}{x} = 1$. This idea will be the basis for L'Hospital Rule.

EXAMPLE 3: Approximate $\sqrt[3]{28}$.

Solution We need to find the closest known value we can compute.

$3 = \sqrt[3]{27}$ so $x_0 = 27$. & we use $f(x) = \sqrt[3]{x}$

Linear approximation: $\cdot\ f'(x) = \frac{1}{3}(x)^{-2/3} = \frac{1}{3x^{2/3}} \rightsquigarrow f'(27) = \frac{1}{3\,3^2} = \frac{1}{27}$

$\cdot\ f(27) = 3$

$$L(x) = f(27) + f'(27)(x-27) = 3 + \frac{1}{27}(x-27) = 2 + \frac{x}{27}$$

$$\sqrt[3]{28} = f(28) \approx L(28) = 2 + \frac{28}{27} = 3 + \frac{1}{27} \cong 3.037\ldots$$

(This assumes that 28 is close enough to 27 for the approximation to be meaningful)

EXAMPLE 4: If the radius of the Earth increases by 1 ft, how much would the surface area increase?

Solution: $A = 4\pi r^2$ $\quad r$ = radius of Earth ≈ 4000 mi

$dr = 1\,\text{ft} = \frac{1}{5280}$ mi.

So $\Delta A \approx dA = A'(r)(dr) = 8\pi r\, dr = 8\pi\, 4000 \cdot \frac{1}{5280}$ mi

$\approx 19.04\ \text{mi}^2$

EXAMPLE 5: The radius of a circular disk is given to be 24 cm, with a maximum measurement error of 0.2 cm. Use differentials to estimate the maximum error in the calculated area of the disk & the relative error

Solution: Area $A(r) = \pi r^2 \rightsquigarrow dA = 2\pi r\, dr$

$\boxed{\text{Error in measurement} = dr}$

- When $r=24$, we know $dr \pm 0.2$ at most, so

$$dA = \pm 2\pi \cdot 24 \cdot 0.2 = \pm 9.6\pi$$

- Maximum error $= 9.6\pi\ cm^2$. when $A = \pi \cdot 24^2$

- Relative error = ?

$$\boxed{\text{Rel error} = \frac{\text{Approx. Value} - \text{True Value}}{\text{True Value}}}$$

True Value = $f(x_0)$ Approx Value = $f(x_0 + \Delta x)$

So $\dfrac{f(x_0+\Delta x) - f(x_0)}{f(x_0)} = \dfrac{f'(x_0)\ \Delta x}{f(x_0)} = \boxed{\dfrac{x_0 f'(x_0)}{f(x_0)}}\ \boxed{\dfrac{\Delta x}{x_0}}$

Condition Number Relative error for measurement

In our example $\dfrac{dA}{A} = \dfrac{9.6\pi}{24^2\pi} = \dfrac{1}{60}$ & $\dfrac{dr}{r} = \dfrac{0.2}{24} = \dfrac{1}{120}$

Condition number is $2 = \frac{1}{60} / \frac{1}{120}$