Lecture XIX: §5.3 Indefinite integrals & Substitution
§5.4 Differential Equations. Separation of variables

§1 Antiderivatives:

Up to now, we've discussed rules to find f' given a function f.

TODAY: We want to reverse the process ("antiderivation"), meaning "Given $f_{(x)}$, can we find $F(x)$ with $F' = \frac{dF}{dx} = f(x)$?"

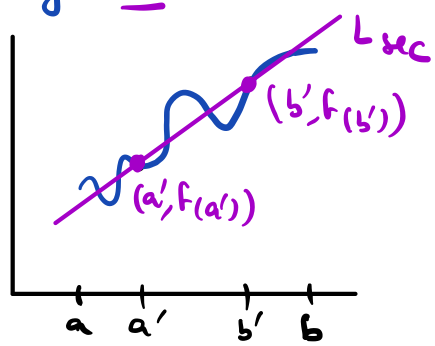
Examples: ① $\frac{dF}{dx} = 3x^2 \rightsquigarrow F = x^3, \ x^3+1, \ldots$

② $\frac{dF}{dx} = \cos x \rightsquigarrow F = \sin(x), \ \sin(x) - \frac{\pi}{4}\sqrt{2}, \ldots$

Guiding question: Can we recover F from f? Almost! (up to an additive constant)

Proposition If $F_{(x)}$ & $G_{(x)}$ are two functions with the same derivative $(=f_{(x)})$ on an interval (a,b) with $a<b$, then $F(x) - G(x)$ is constant on (a,b).

Q Why? A Mean Value Theorem. Pick a', b' with $a<a'<b'<b$



Since $H_{(x)} = F_{(x)} - G_{(x)}$ is differentiable on (a,b) we know
- H is continuous on $[a',b']$
- H is differentiable on (a',b')

By MVT, we can find c in (a',b') with

$$H'(c) = \text{slope of } L_{sec}$$

But $H'(c) = 0$ so L_{sec} is a horizontal line. This means $H(a') = H(b')$ for any pair a', b' in (a,b).

Conclude: $H = F-G$ is a constant function.

Consequence: Antiderivatives are not unique, but they all differ by constants.

Notation $F(x) + C = \int f(x)\,dx$ We call it the antiderivative or indefinite integral of f.

Remark: Every formula for derivatives has an analog for indefinite integrals. This is based on "Derivative rules for differentials".

§2. Differentiation Formulas in differential notation:

Fix u, v function of 1 variable x.

(1) Power Rule: $y = u^n \rightsquigarrow dy = n u^{n-1}\, du$

(2) Product Rule: $y = uv \rightsquigarrow dy = d(uv) = u\,dv + v\,du$

[Q: Why? A: $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ & multiply both sides by dx]

(3) Quotient Rule: $y = \frac{u}{v} \rightsquigarrow dy = d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$

(4) Chain Rule: $y = f(u)$ $u = g(x)$ $\rightsquigarrow dy = f'(u)\,du$

$du = g'(x)\,dx$

so $dy = f'(u)\, g'(x)\, dx = f'(g(x))\, dx$.

§3. Rules for Antiderivatives

(1) Power Rule: $\frac{d x^{m+1}}{dx} = (m+1)x^m \rightsquigarrow d x^{m+1} = (m+1) x^m dx \rightsquigarrow x^m dx = \frac{d x^{m+1}}{m+1}$ if $m \neq -1$

So $\boxed{\int x^{m+1}\, dx = \frac{x^{m+1}}{m+1} + C \quad \text{as long as } m \neq -1}$

(2) Trigonometric Rules:

$\int \sin x\, dx = -\cos x + C$ $\qquad \int \cos x\, dx = \sin x + C$

$\int \sec^2 x\, dx = \tan x + C$ $\qquad \int \csc^2(x)\, dx = -\cot(x) + C$

$\int \tan x \sec x\, dx = \sec x + C$ $\qquad \int \cot(x)\csc(x) = -\csc(x) + C$

(3) Multiplication by constants & additivity:

(1) a constant $\int a f(x)\, dx = a \int f(x)\, dx$

(2) $\int (f(x) + g(x))\, dx = \int f(x)\, dx + \int g(x)\, dx$.

⇝ We get antiderivatives for polynomials (in fractional powers of x without $\frac{1}{x}$)

Examples • $\int 3x^2+4x+2\,dx = 3\int x^2 dx + 4\int x\,dx + 2\int 1\,dx$

$= 3\left(\frac{x^3}{3}+C_1\right) + 4\left(\frac{x^2}{2}+C_2\right) + 2(x+C_3)$

$= x^3 + 2x^2 + 2x + C \qquad C$ constant.

• $\int x^{1/3}(x+2)^2\,dx = \int x^{1/3}(x^2+4x+4)\,dx = \int x^{7/3}+4x^{4/3}+4x^{1/3}dx$

$= \dfrac{x^{10/3}}{10/3} + 4\dfrac{x^{7/3}}{7/3} + 4\dfrac{x^{4/3}}{4/3} + C = \frac{3}{10}x^{10/3} + \frac{12}{7}x^{7/3} + 3x^{4/3} + C$

4 Substitution (Most subtle rule = counterpart to chain rule).

IDEA: Reverse of chain rule

Example: $\int \sqrt{x^2+1}\;2x\,dx = ?$

Call $u = x^2+1$. Then $du = 2x\,dx$

$$\int \sqrt{x^2+1}\;2x\,dx = \int \sqrt{u}\,du = \int u^{1/2}\,du = \frac{u^{3/2}}{3/2} + C$$

To finish substitute back u:

$$\int \sqrt{x^2+1}\;2x\,dx = \frac{2}{3}(x^2+1)^{3/2} + C.$$

Substitution Rule: If $F(x) = h(u(x))$, write $F = h(u)$ & $u = u(x)$. Then $dF = h'(u)\,du$ & $du = u'(x)\,dx$

Conclusion $dF = h'(u)\,u'(x)\,dx$ &

$$\int h'(u(x))\,\underbrace{u'(x)\,dx}_{=du} = \int h'(u)\,du = h(u) + C = h(u(x)) + C = F(x) + C$$

Q: How do we use this to compute $\int f(x)\,dx$

1. Recognize part of $f(x)\,dx$ as $u'\,dx$
2. Substitute u, integrate $\int h'(u)\,du$ & substitute back to express the answer as a function of x.

Notes: • Integration involves 2 substitutions

• We can check our answer is correct using the Chain Rule

• The Rule is straight forward if we view the differentiation step as computing a differential.

• Finding the expression $u = u(x)$ can be tricky! (Practice helps!) We must be able to write the integrand $f(x)\,dx$ as $h(u)\,du$, making sure NO x remains!

EXAMPLES: ① $\displaystyle\int \cos^5 x \sin x \, dx \underset{\substack{u = \cos x \\ du = -\sin x\, dx}}{=} \int u^5(-du) = -\frac{u^6}{6} + C = -\frac{\cos^6 x}{6} + C$

② $\displaystyle\int \sin(5x)\,dx \underset{\substack{u=5x \\ du = 5dx}}{=} \int \sin u \frac{du}{5} = \frac{1}{5}(-\cos u) + C = \frac{-\cos(5x)}{5} + C$

③ $\displaystyle\int \frac{dx}{(x-7)^7} \underset{\substack{u = x-7 \\ du = dx}}{=} \int \frac{du}{u^7} = \int u^{-7}\,du = \frac{u^{-6}}{-6} + C = \frac{-1}{6}\frac{1}{(x-7)^6} + C$

§4. Differential Equations & separation of variables:

The theory of differential equations aims to recover a function from a relation among its higher-order derivatives

Examples: ① $y' = 2x$. By integration $y = \int dy = \int 2x\,dx = x^2 + C$

② $y'' = -y$ or $y'' + y = 0$

By inspection $y(x) = \sin x$ & $y = \cos y$ are solutions

HARD: Show all solutions are of the form $y = a\sin x + b\cos x$ for fixed parameters a, b.

③ $y''' + x^2 y'' - y = 0 \rightsquigarrow$ Soln = ???

• Certain simple O.D.E.s (ordinary differential equations) can be solved via the method of separation of variables (analog of implicit differentiation)

Example $y' = -x^2y^2$

Write $\dfrac{dy}{dx} = -x^2y^2$ or $dy = -x^2y^2dx$

STEP 1: We group different variables on different sides of the equation ("separation step")

$$-\frac{dy}{y^2} = x^2dx$$

STEP 2: Integrate both sides & put the constant C only on the x-side

$$\frac{1}{y} = \int -\frac{dy}{y^2} = \int x^2dx = \frac{x^3}{3} + C \qquad C = \text{arbitrary constant}$$

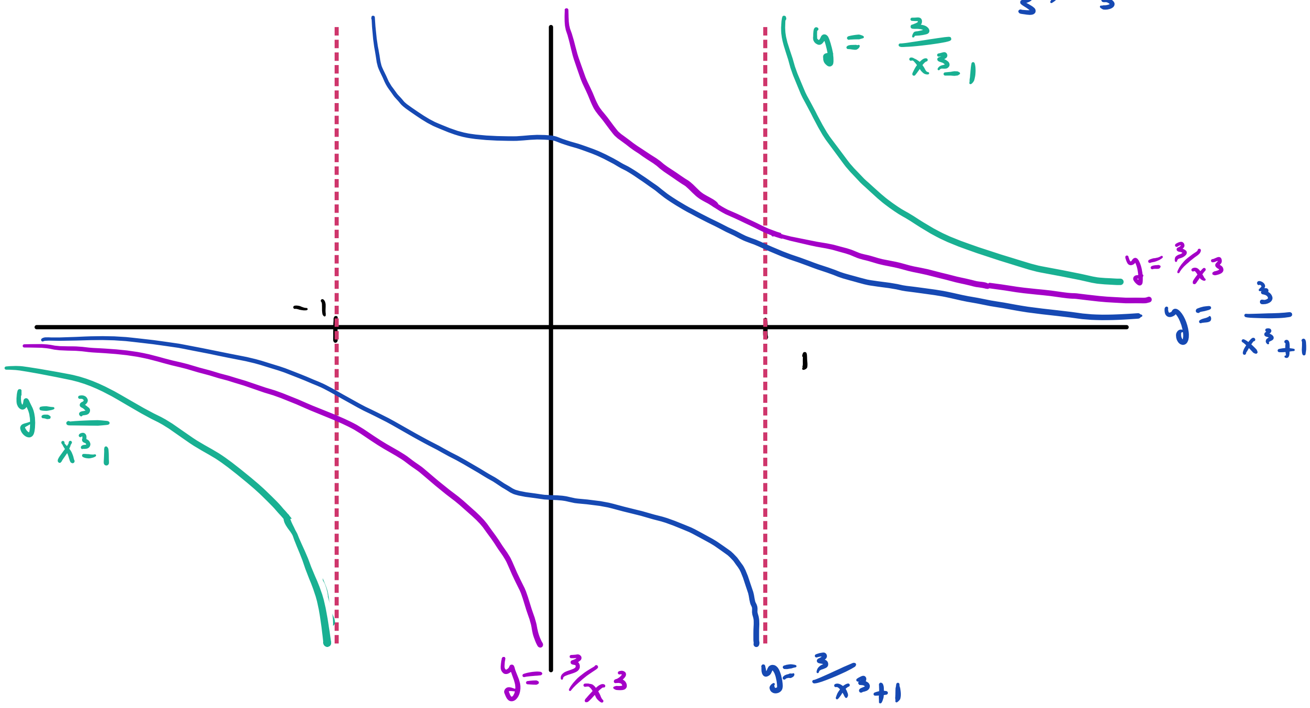
STEP 3: Solve for y

General Solution: $\boxed{y = \dfrac{1}{\frac{x^3}{3} + C}}$

Note: We get a 1-parameter family of solutions (C = parameter)

Finding a particular solution means to choose a value for C. Often, we do this by specifying initial conditions ($y(0)$, $y'(0)$, $y''(0)$, ... etc.), typically as many as the number of parameters.

Examples Draw 3 particular solutions ($C = \frac{1}{3}, -\frac{1}{3}, 0$)



Note: Separation of variables only works for very special ODEs, namely if

- equation only involves y', y, x (1st order)
- need to split the equation into $g(y)\,dy = f(x)\,dx$

Non-example: $y' = \dfrac{x+y}{x-y}$

Example: Solve $\begin{cases} y' = 2y^2(4x^3+4x^{-3}) \\ y(0) = 1 \quad \text{(initial condition)} \end{cases}$

Solution: $\dfrac{dy}{y^2} = (8x^3+8x^{-3})\,dx$ so integration gives:

$$-\frac{1}{y} = \int \frac{dy}{y^2} = \int (8x^3+8x^{-3})\,dx = \frac{8}{4}x^4 + \frac{8}{-2}x^{-2} + C$$
$$= 2x^4 - 4x^{-2} + C$$

So $y_{(x)} = \dfrac{1}{-2x^4+4x^{-2}+C}$ (General solution)

Particular solution $1 = y(0) = \dfrac{1}{C}$ so $C = 1$

Solution is $\boxed{y(x) = \dfrac{1}{-2x^4+4x^{-2}+1}}$