Lecture XX: §5.5 Motion under gravity

Recall $s(t) :=$ position of a particle with respect to time $(t \geq 0)$

$v(t) := s'(t) =$ velocity (speed $= |v(t)|$)

Newton's Law of Motion:

(I) A particle in a state of rest or motion will continue to be so unless an external force is applied to it.

(II) $a(t) =$ acceleration $= s''(t)$
$m =$ mass
$F =$ force

Then: $a(t) = \frac{F}{m}$, equivalently

$$\boxed{F = ma = m\,s''(t)} \quad (*)$$

(ordinary differential equation)

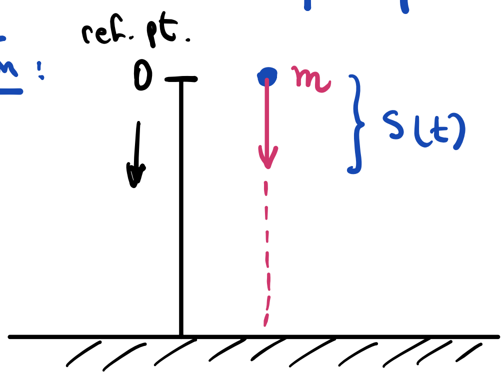
(III) To every action, there's an equal and opposite reaction.

GOAL: Solve $(*)$. Since $s''(t)$ is involved, we'll need 2 initial conditions, typically $s(0)$ & $s'(0)$, to give a particular solution to $(*)$

§1. Examples: Gravity force induces a constant acceleration $g \approx 9.8\frac{m}{s^2} = 32\frac{ft}{s^2}$

PROBLEM 1: Find the motion of a stone of mass m which is dropped from a point above the surface of the Earth.

Solution:



Initial conditions: $s(0) = 0$ (reference pt)

$v(0) = 0$ (dropped so no velocity)

Equation $m\,a(t) = F = mg$

gravity force, no other forces acting (no air resistance, etc.)

$\rightsquigarrow s''(t) = a(t) = g$ constant

Solve for s by integrating twice (don't forget the constants!)

- $v'(t) = a(t)$ so $v(t) = \int g\,dt = gt + C_1$

Initial condition $v(0)=0$ yields $0=0+C_1$, so $C_1=0$ & $v(t)=gt$

- $s'(t)=v(t)$ so $s(t)=\int gt\,dt = g\frac{t^2}{2}+C_2$

Initial condition $s(0)=0$ yields $0=0+C_2$, so $C_2=0$ & $s(t)=g\frac{t^2}{2}$

$$\boxed{\text{Solution: } s(t)=\tfrac{1}{2}gt^2}$$

General Solution $s(t)=\frac{1}{2}gt^2+C_1t+C_2$ for C_1, C_2 constants

These are determined by fixing 2 "independent" conditions.

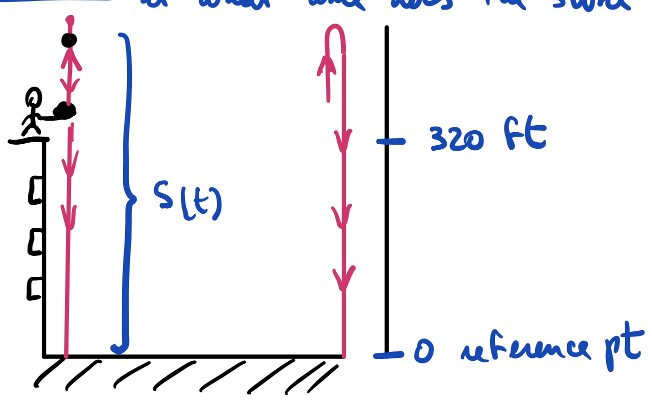
PROBLEM 2: Assume a stone is thrown upwards at 128 ft/s from the roof of a building 320 ft high.

(1) Find its trajectory

(2) Determine its maximal height

(3) ______ at what time does the stone reach the ground.

Solution:



Initial conditions $s(0)=320$

$s'(0)=128$

(thrown upwards, so sign is +)

Equation: $m\,a''(t)=F=-mg$

gravity points in opposite direction to height increase.

$\rightsquigarrow s''(t) = a(t) = -g$ constant (-32)

Again, we solve for $s(t)$ by integrating twice:

$$v(t) = \int s''(t)\,dt = \int -g\,dt = -gt+C_1$$

$$s(t) = \int v(t)\,dt = \int(-gt+C_1)\,dt = -g\frac{t^2}{2}+C_1t+C_2$$

(1) Use initial conditions to find C_1 & C_2.

$s(0) = 320 = C_2$ & $s'(0)=128=C_1$

$\rightsquigarrow s(t) = -16t^2 + 128t + 320.$

(2) Maximal height $\Rightarrow$ t is a critical pt of $S(t)$

$v(t) = S'(t) = 0$ so $0 = -32t + 128$ we get $t = \frac{128}{32} = 4 > 0$

So Maximum height is reached after 4 seconds.

- $S(4)$ = maximum height $= -16 \cdot 4^2 + 128 \cdot 4 + 320 = \boxed{576 \text{ ft}}$

(3) Hit the ground means $S(t) = 0$. We get a quadratic equation in t that we can easily solve:

$$-16t^2 + 128t + 320 = 0 \implies t = \frac{-128 \pm \sqrt{128^2 + 4 \cdot 16 \cdot 320}}{2 \cdot (-16)}$$

We get $t = 10$ & $t = -2$ as solutions

We only keep the one with $t > 0$.

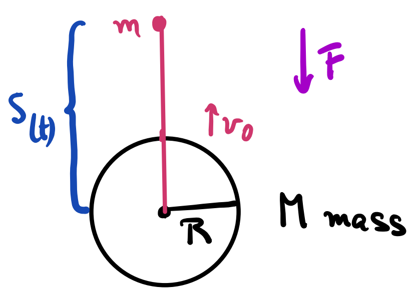
Conclude: The rock hits the ground after 10 seconds.

Q: What's its speed? A: $|v(10)| = |-32 \cdot 10 + 128| = |-192| = 192 \frac{\text{ft}}{\text{s}}$

Summary:
- Set up the initial conditions
- Equation becomes $S''(t) = \pm g$
- Integrate twice to solve for S & use initial conditions to determine the constants C_1 & C_2

§2. Escape Velocity:

GOAL: Determine the initial velocity v_0 a rocket fired vertically should have to come to rest & escape completely from the Earth's gravitational attraction



Assumption: Earth is viewed as a particle of mass M located at its center

When firing the rocket, the gravitational force will depend on the distance from the rocket to the center of the Earth.

Newton's Law of Gravitation: Any 2 particles of matter attract each other with a force that is jointly proportional to their masses & inversely proportional to the square of the distance between them.

$$\boxed{F = -G\,\frac{M\,m}{s^2}}$$

$G>0$ constant, $s=$ distance
M, m masses

Sign is $-$ because the attraction is opposite to the direction of movement.

Using this, we get $\underset{\text{(rocket's perspective)}}{m\,a(t)} = \underset{\text{(gravitational force)}}{-\dfrac{G\,M\cdot m}{s(t)^2}}$ $\rightsquigarrow$ $\boxed{s''(t) = \dfrac{-GM}{s^2(t)}}$ (*)

Note: On the ground $s''(0) = -g$ & $s(0) = R$ (= radius of the Earth)

From this we get $-g = \dfrac{-GM}{R^2}$ at $t=0$, giving $\boxed{GM = gR^2}$

- Next, we replace this in (*) to get: $\boxed{v'(t) = s''(t) = -g\,\dfrac{R^2}{s^2(t)}}$

Remember: We want to solve for $v(t)$!

Escape velocity = initial velocity $v(0) = v_0$ TO BE DETERMINED!

Trick: Think of s as a variable ($v = v(s)$) and use chain rule:

$$\frac{dv}{dt} = \frac{dv}{ds}\,\frac{ds}{dt} = \frac{dv}{ds}\cdot v.$$

So we get $\dfrac{dv}{ds}\cdot v = -g\,\dfrac{R^2}{s^2}$ $\rightsquigarrow$ $\underset{(v\text{-side})}{v\,dv} = \underset{(s\text{-side})}{-g\,\dfrac{R^2}{s^2}\,ds}$

The variables are separated & we can solve by integrating

$$\int v\,dv = \int -g\,\frac{R^2}{s^2}\,ds \rightsquigarrow \frac{v^2}{2} = g\,\frac{R^2}{s} + C$$

Q: How to pick C? $v(0) = v_0$ & $s(0) = R$ so at $t=0$

we get: $\frac{v_0^2}{2} = g\frac{R^2}{R} + C \rightsquigarrow C = \frac{v_0^2}{2} - gR$

Solution $$\boxed{\frac{v^2}{2}(t) = g\frac{R^2}{S(t)} + \left(\frac{v_0^2}{2} - gR\right)}$$

Q: How to escape gravitational force?

A We need $v(t) > 0$ for all t

Note $g\frac{R^2}{S(t)} \xrightarrow[t\to\infty]{} 0^+$ if we escape the Earth (since $S(t) \to \infty$)

So the only way to ensure $v(t) > 0$ is to require $\frac{v_0^2}{2} - gR \geq 0$

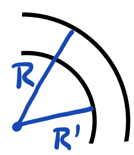
Since $v_0 > 0$, this gives $\boxed{v_0 \geq \sqrt{2gR} = \text{Escape velocity}} = \sqrt{\frac{2GM}{R}}$

Note This formula works for any plane (g = acceleration due to gravity on that plane)

Values for Earth: $R = 4000$ mi

$g = 32\frac{ft}{s^2} = \frac{32}{5280}\frac{mi}{s^2}$ so $\sqrt{2gR} \simeq 7\frac{mi}{s}$

Applications ① If the mass M is preserved but R decreases to R', then the escape velocity increases! $\left(= \sqrt{\frac{2GM}{R'}} > \sqrt{\frac{2GM}{R}} \right)$



② If the escape velocity > speed of light, the light can never escape the gravitational force. This explains black holes!