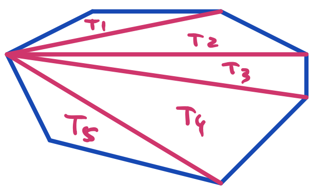
Lecture XXI: §6.2 The Problem of areas

GOAL: Find a method to compute areas of regions enclosed by simple curves.

§1. The Problem of areas:

EXAMPLE 1: Closed convex polygons

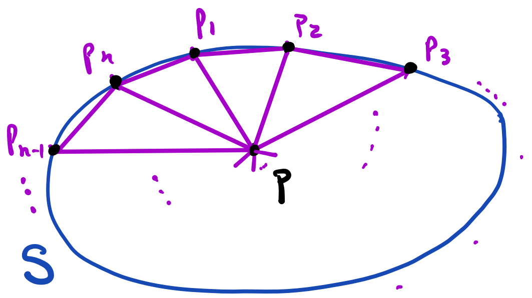


1. Triangulate it from one vertex
2. Add up the areas of the triangles

Triangles = # sides − 2

Q: What about non-linear convex regions?

A: Method of exhaustion.



1. Pick a point P in its interior
2. Pick a finite number of points in the boundary say, $P_1, \ldots, P_n$ (clockwise)
3. Build all triangles $\overset{\triangle}{PP_1P_2}, \overset{\triangle}{PP_2P_3}, \ldots, \overset{\triangle}{PP_{n-1}P_n}$ & $\overset{\triangle}{PP_nP_1}$

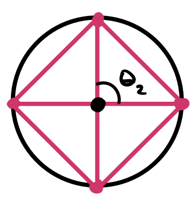
Note: • Convexity ensures all $\triangle$s are in the interior of the region!

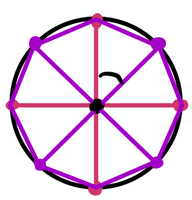
• Triangles do NOT overlap (only meet along edges/vertices)

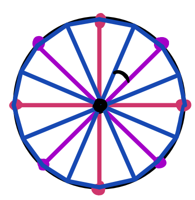
4. Make $n \to \infty$ & place the points $P_1, \ldots, P_n$ close to each other equidistributed along the border of the region S The Triangles will end up covering almost all the region S, so they approximate Area(S).

EXAMPLE 2: S = circle of radius R, $P = (0,0)$

Choose $P_1, \ldots, P_{2^N}$ as the vertices of a regular 2^N-gon, called S_N.







$n = 2^N$

S_2 $\qquad \theta_2 = \frac{\pi}{2}$

S_3 $\qquad \theta_3 = \frac{\pi}{4}$

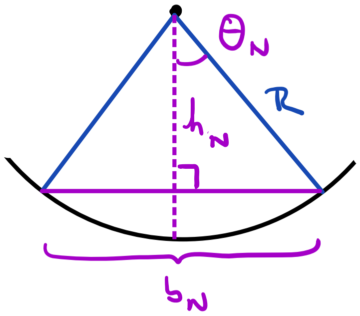
S_4 $\qquad \theta_4 = \frac{\pi}{8}$

Q: Why 2^n? A: 1. At each step, we divide the angle θ in half

$$\rightsquigarrow \theta_N = \frac{2\pi}{2^N} = \frac{\pi}{2^{N-1}}$$

2. At each step, S_N is contained in S_{N+1} (square $\subseteq$ octagon $\subseteq$ ...)

3. Easy area calculation: $\text{Area}(S_N) = 2^N$ Area of 1 triangle.



$$\text{Area } \Delta = \frac{h_N \cdot b_N}{2}$$

Conclude: $\text{Area}(S_N) = 2^N \dfrac{h_N\, b_N}{2} = \dfrac{h_N}{2}(2^N b_N)$
$= \dfrac{h_N}{2}\ \text{Perimeter}(S_N)$

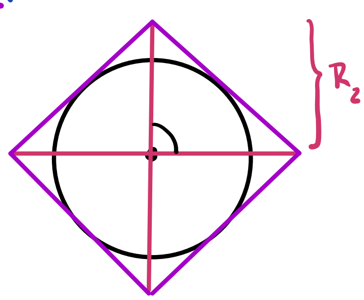
Q: What happens as we let $N \to \infty$? (ie, as we exhaust the circle)

$$h_N = R\cos\theta_N = R\cos\left(\frac{\pi}{2^{N-1}}\right) \xrightarrow[N\to\infty]{} R = \text{radius of the circle}$$

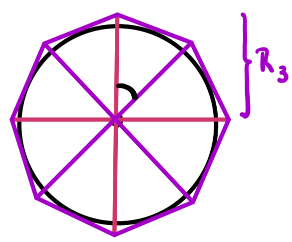
$$\text{Perimeter}(S_N) \xrightarrow[N\to\infty]{} \text{Perimeter of the circle} = \text{circumference} = 2\pi R$$

$$\text{So Area(circle)} \overset{\triangle!}{=} \lim_{N\to\infty} \text{Area}(S_N) = \frac{R}{2}\, 2\pi R = \boxed{\pi R^2}$$

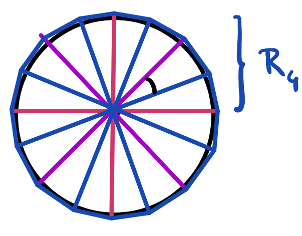
Q: What if we had taken circumscribed 2^N-gons rather than inscribed ones?



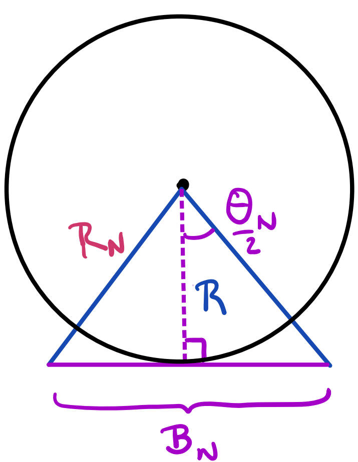
Outer S_2
$\theta_2 = \frac{\pi}{2}$



Outer S_3
$\theta_3 = \frac{\pi}{4}$



Outer S_4
$\theta_4 = \frac{\pi}{8}$



Note: 1. Tangencies occur at the midpoints of each side of the 2^N-gon

2. Outer side has length B_N & $2R\tan\frac{\theta_N}{2} = B_N$

3. $\text{Area}(\text{Outer } S_N) = 2^N \cdot \dfrac{R\, B_N}{2} = \dfrac{R}{2}(2^N B_N)$
$= \dfrac{R}{2}\ \text{Perimeter}(\text{Outer } S_N)$

Again: $R_N \xrightarrow[N\to\infty]{} R$ $\quad$ $\text{Perimeter}(\text{Outer } S_N) \xrightarrow[N\to\infty]{} 2\pi R$

We get $\boxed{\text{Area}(S_N) \leq \text{Area}(\text{Circle}) \leq \text{Area}(\text{Outer } S_N)}$ (*)

$$\frac{h_N}{2}\,\text{Perim}(S_N) \leq \text{Area}(\text{Circle}) \leq \frac{R}{2}\,\text{Perimeter}(\text{Outer } S_N)$$

$$\frac{h_N}{2}\,\text{Perim}(S_N) \xrightarrow{N\to\infty} R^2\pi \qquad \frac{R}{2}\,\text{Perimeter}(\text{Outer } S_N) \xrightarrow{N\to\infty} R^2\pi$$

By "Squeeze Lemma": $\text{Area}(\text{Circle}) = \pi R^2$ (this justifies $\triangle$)

<u>Obs</u>: 1. We get the formula for Area(Circle) from the Perimeter(Circle).

2. We can use this method to approximate π to any digit we want!

<u>Q</u> How to approximate π?

Set $R=1$. Use $h_N = \cos\frac{\pi}{2^N}$,

$b_N = 2\sin\frac{\pi}{2^N} \rightsquigarrow$ Inner Perimeter $2^{N+1}\sin\left(\frac{\pi}{2^N}\right)$

$B_N = 2\tan\frac{\pi}{2^N} \rightsquigarrow$ Outer Perimeter $2^{N+1}\tan\left(\frac{\pi}{2^N}\right)$

So Outer Area $= 2^N\tan\left(\frac{\pi}{2^N}\right)$

Inner Area $= h_N\,\frac{\text{Inner Perim}}{2} = \cos\frac{\pi}{2^N}\,\frac{2^{N+1}}{2}\sin\frac{\pi}{2^N} = 2^N\sin\frac{\pi}{2^N}\cos\frac{\pi}{2^N}$

$\underset{\downarrow}{=} 2^{N-1}\sin\left(\frac{\pi}{2^{N-1}}\right)$

$\sin(2x) = 2\sin x\cos x$ for $x = \frac{\pi}{2^N}$

Replace in (*): $\text{Area}(S_N) \leq \text{Area}(\text{Circle}) \leq \text{Area}(\text{Outer } S_N)$

$$\overset{\shortparallel}{2^{N-1}\sin\frac{\pi}{2^{N-1}}} \leq \pi \leq 2^N\tan\frac{\pi}{2^N}$$

If the ends agree in the first k decimal places (for N large enough), then these are also the first k digits of π.

Q Why inscribed vs circumscribed? We'll use inner & outer approximations to compute areas under the graph of a positive functions, replacing $\triangle$s with rectangles. This will lead naturally to "Riemann Sums" defining integrals. $A = \int_a^b f(x)\,dx =$