Lecture XXII: §6.3 Summation notation & some sums
§6.4 The area under a curve. Definite Riemann integrals

Last Time: we computed areas of convex regions by exhaustion (covering them with triangles)

TODAY: Compute areas under curves defined by continuous functions using rectangles (Riemann Sums).

- 2 notations for sums: S $\to$ Σ (Greek S) = discrete sums; $\int$ (Integral sign) = "continuous sums"

§1. Summation Σ:

Notation: Given numbers $a_1, \ldots, a_n$, we write $a_1 + \cdots + a_n = \sum_{j=1}^{n} a_j$

Examples:

$$\sum_{j=1}^{5} j = 1+2+3+4+5 = 15 = \frac{5\cdot 6}{2}$$

$$\sum_{k=2}^{7} k^2 = 4+9+16+25+36+49 = 139 = \frac{7\cdot 8\cdot 15}{6} - 1$$

$$\sum_{i=1}^{4} \left((-1)^i \frac{1}{i}\right) = (-1)^1\frac{1}{1} + (-1)^2\frac{1}{2} + (-1)^3\frac{1}{3} + (-1)^4\frac{1}{4}$$

$$= -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} = \frac{-7}{12}$$

GOAL: Find closed formulas for

$$\sum_{i=1}^{n} i = 1+2+\cdots+n$$

$$\sum_{k=1}^{n} k^2 = 1^2+2^2+\cdots+n^2$$

$$\vdots$$

This formulas will help us compute Riemann Sums next time.

Proposition 1: $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

Why? This argument is due to Gauss.

$$\begin{array}{cccccccccc} & \boxed{1} & + & \boxed{2} & + & \cdots & + & \boxed{(n-1)} & + & \boxed{n} \\ + & n & + & (n-1) & + & \cdots & + & 2 & + & 1 \\ \hline & n+1 & + & n+1 & + & \cdots & + & n+1 & + & n+1 \end{array}$$

& Add "column by column"

$= (n+1)\cdot \#\text{columns} = (n+1)\,n$

Conclude $(1+2+\cdots+(n-1)+n) + (n+(n-1)+\cdots+1) = (n+1)n$

So $2(1+2+\cdots+n) = 2\sum_{k=1}^{n} k = (n+1)n$

$$\boxed{\sum_{k=1}^{n} k = \frac{(n+1)n}{2}}$$

- This is a very elegant argument, but it won't help us with finding $\sum_{k=1}^{n} k^2$, $\sum_{k=1}^{n} k^3$, etc. For this, we give a second way of finding the formula for $\sum_{k=1}^{n} k$.

- **2^{nd} approach** Use telescopic sums

Recall: $(k+1)^2 = k^2+2k+1$, so $\boxed{(k+1)^2-k^2 = 2k+1.}$ $(*)$

Q: What happens if we sum $(*)$ as we vary k between 1 and n?

A: A lot of cancellations occur!

only survivors!

$$\sum_{k=1}^{n}\left((k+1)^2-k^2\right) = (\cancel{2^2}-1^2)+(\cancel{3^2}-\cancel{2^2})+\cdots+(\cancel{n^2}-\cancel{(n-1)^2})+((n+1)^2-\cancel{n^2})$$

$$= (n+1)^2-1^2$$

In general, this will lead to a telescopic sum:

from last term

$$(a_2-a_1)+(a_3-a_2)+\cdots+(a_{n+1}-a_n) = \sum_{k=1}^{n}(a_{k+1}-a_k) = a_{n+1}-a_1$$

from first term

Now, we can sum this via the right-hand side of $(*)$:

$$\sum_{k=1}^{n}(2k+1) = \begin{array}{c|c|c} 2\cdot 1 & + & 1 \\ 2\cdot 2 & + & 1 \\ 2\cdot 3 & + & 1 \\ \vdots & & \\ 2\cdot n & + & 1 \\ \hline \multicolumn{3}{c}{① + ②} \end{array}$$

Again, we "add column-by-column"

$① = 2\cdot 1+2\cdot 2+\cdots+2\cdot n = 2(1+2+\cdots+n)$
$= 2\sum_{k=1}^{n} k$

$② = \underbrace{1+1+\cdots+1}_{n \text{ times}} = n$

So we get $\displaystyle\sum_{k=1}^{n}(2k+1) = 2\boxed{\sum_{k=1}^{n} k} + n$

WHAT WE WANT!

by (*) $\parallel$

$$\sum_{k=1}^{n}\left((k+1)^2-k^2\right) = (n+1)^2-1 = n^2+2n+1-1 = n(n+2)$$

Conclude:

$$2\left(\sum_{k=1}^{n} k\right) + n = n(n+2)$$

$$2\left(\sum_{k=1}^{n} k\right) = n(n+2)-n = n(n+1)$$

$$\boxed{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}}$$

This approach can be generalized to compute $\displaystyle\sum_{k=1}^{n} k^2$, $\displaystyle\sum_{k=1}^{n} k^3$ (HW5)

Proposition 2: $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

Why? We use a different telescopic sum.

$$\boxed{(k+1)^3-k^3 = k^3+3k^2+3k+1-k^3 = 3k^2+3k+1.} \quad (**)$$

So $\displaystyle\sum_{k=1}^{n}\left((k+1)^3-k^3\right) = (\cancel{2^3}-1^3)+(\cancel{3^3}-\cancel{2^3})+\cancel{\ }\cdots\cancel{+}\,((n+1)^3-\cancel{n^3}) = (n+1)^3-1 = n^3+3n^2+3n$

only survivors

telescopic!

On the other hand $\displaystyle\sum_{k=1}^{n}(3k^2+3k+1) =$

$$\begin{array}{ccccc} 3\cdot1^2 & + & 3\cdot1 & + & 1 \\ + \; 3\cdot2^2 & + & 3\cdot2 & + & 1 \\ \vdots & \vdots & & & \\ 3\cdot n^2 & + & 3\cdot n & + & 1 \\ \hline (1) & + & (2) & + & (3) \end{array}$$

Add columns one at a time

where (1) $= 3\cdot1^2+3\cdot2^2+\cdots+3\cdot n^2 = 3(1^2+2^2+\cdots+n^2) = 3\boxed{\displaystyle\sum_{k=1}^{n} k^2}$

WHAT WE WANT!

(2) $= 3\cdot1+3\cdot2+\cdots+3\cdot n = 3(1+2+\cdots+n) = 3\displaystyle\sum_{k=1}^{n} k$

$= 3\,\dfrac{(n+1)n}{2}$ (by Prop 1)

(3) $= \underbrace{1+1+\cdots+1}_{n \text{ times}} = n$

So we get $n^3+3n^2+3n = \sum_{k=1}^{n}((k+1)^3-k^3) \underset{\text{by}(**)}{=} \sum_{k=1}^{n}(3k^2+3k+1) =$

$$= 1 + 2 + 3 = 3\boxed{\sum_{k=1}^{n} k^2} + 3\,\frac{(n+1)n}{2} + n$$

WHAT WE WANT!

So $3\sum_{k=1}^{n} k^2 = n^3+3n^2+3n - 3\,\frac{(n+1)n}{2} - n$

$$= n^3+3n^2+3n - \frac{3n^2}{2} - \frac{3n}{2} - n$$

$$= \frac{1}{2}(2n^3+3n^2+n) = \frac{1}{2}(n(2n^2+3n+1))$$

$$= \frac{1}{2}\, n((2n+1)(n+1))$$

Conclude $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

Proposition 3: $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$ (see HW5)

- General procedure to compute $\sum_{k=1}^{n} k^s$ for $s=1,2,3,\dots$

STEP 1 Use the Binomial Theorem to write $(k+1)^{s+1}-k^{s+1}$.

$$(k+1)^{s+1}-k^{s+1} = (s+1)k^s + \binom{s+1}{2}k^{s-1} + \dots + \binom{s+1}{s}k + 1$$

STEP 2: Add up both sides:

- Left-hand side: telescopic sum:

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$$\sum_{k=1}^{n}(k+1)^{s+1}-k^{s+1} = (\cancel{2^{s+1}} - 1^{s+1}) + (3^{s+1} - \cancel{2^{s+1}}) + \dots + ((n+1)^{s+1} - \cancel{n^{s+1}})$$

$$= (n+1)^{s+1} - 1^{s+1}.$$

- Right-hand side:

$\sum_{k=1}^{n}\left((s+1)k^s + \binom{s+1}{2}k^{s-1} + \dots + \binom{s+1}{s}k + 1\right) =$ arrange in columns & add them up

$$= (s+1)\boxed{\sum_{k=1}^{n} k^s} + \binom{s+1}{2}\boxed{\sum_{k=1}^{n} k^{s-1}} + \dots + \binom{s+1}{s}\boxed{\sum_{k=1}^{n} k^1} + n\cdot 1$$

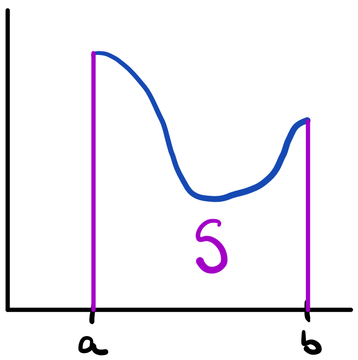
WHAT WE WANT!

formulas for lower exponents are assumed to be known!

So we can compute $\sum_{k=1}^{n} k^s$ using this approach!

§2. The area under a curve

GOAL: given a continuous function $f: [a,b] \longrightarrow \mathbb{R}_{\geq 0}$, (i.e. $\{x \geq 0\}$) we want to find the area of the region S that lies under the curve $y = f(x)$ & above the x-axis



We use 2 approximations to compute Area(S)

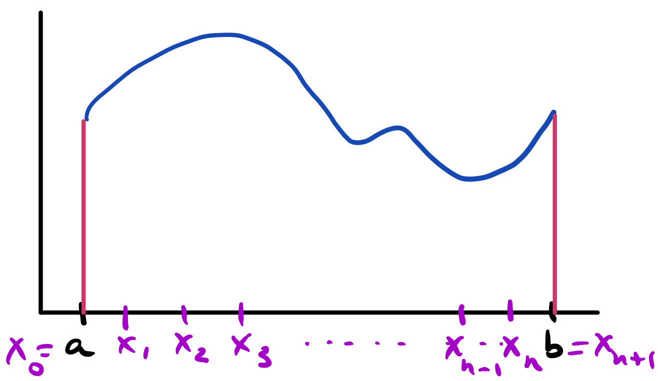
① Overestimate with larger rectangles (in trap S)
⇝ Upper Riemann Sums

② Underestimate with small rectangles (exhaust S)
⇝ Lower Riemann Sums

Both types of rectangles will have:

- base on the x-axis of small length
- height: either a max (for ①) or a min (for ②) of f restricted to each base

STEP 1 Pick points $x_1, \ldots, x_n$ that determine the bases of these $\square$



We write the differences as
$$\begin{aligned} \Delta x_1 &= x_1 - a \\ \Delta x_2 &= x_2 - x_1 \\ &\vdots \\ \Delta x_n &= x_n - x_{n-1} \\ \Delta x_{n+1} &= b - x_n \end{aligned}$$

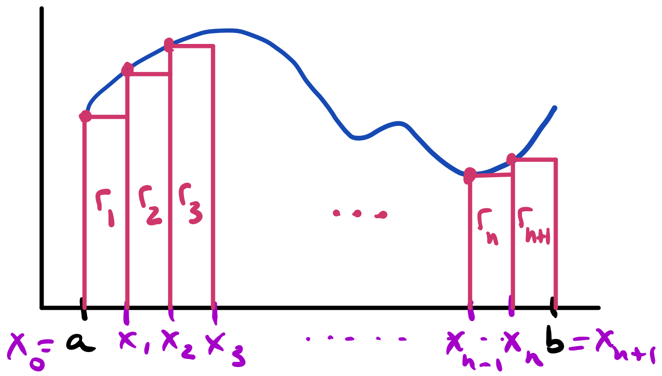
we assume they are all small (which means n is going to be larger!)

To fix ideas, we can make $\Delta x_k = \dfrac{b-a}{n+1}$ for all $k = 1, 2, \ldots, n+1$.

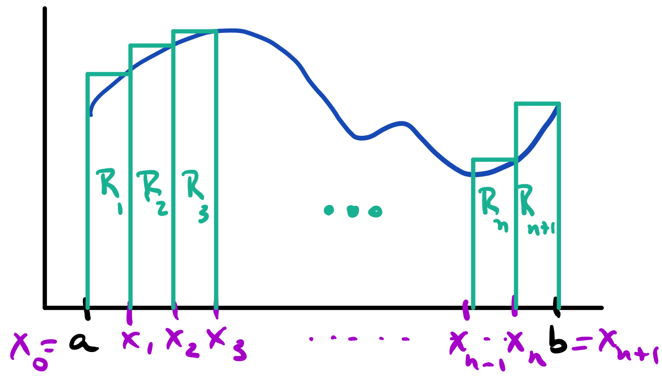
STEP 2: Use these intervals as the bases of our rectangles.

$n+1$ small rectangles $= r_1, \ldots, r_{n+1}$ & height of r_k = min of f on $[x_{k-1}, x_k]$

$n+1$ large rectangles $= R_1, \ldots, R_{n+1}$ & height of $R_k = \max$ of f in $[x_{k-1}, x_k]$



LOWER SUMS



UPPER SUMS

If f is continuous $m_k = $ height of $r_k = f(\underline{x}_k^*)$ for some $\underline{x}_k^*$ in $[x_{k-1}, x_k]$

$M_k = $ height of $R_k = f(\overline{x}_k^*)$ —— $\overline{x}_k^*$ in $[x_{k-1}, x_k]$

The existence of $\underline{x}_k^*$ & $\overline{x}_k^*$ is garanteed by the Extreme Values Thm.

STEP 3: We compare S with the areas covered by small & larger rectangles:

$$\boxed{A(r)} := \sum_{k=1}^{n+1} \text{Area}(r_k) \leq \text{Area}(S) \leq \sum_{k=1}^{n+1} \text{Area}(R_k) := \boxed{A(R)}$$

As we increase n & we decrease all Δx_k (bases of the rectangles r_k & R_k), the heights $f(\underline{x}_k^*)$ & $f(\overline{x}_k^*)$ get closer to each other. This gives:

Theorem: If f is continuous, $\sum_{k=1}^{n} \text{Area}(r_k)$ & $\sum_{k=1}^{n+1} \text{Area}(R_k)$ have the same limit as $n \to \infty$. This is the value of Area(S)

Why? Since f is continuous, the EVT gave us $f(\overline{x}_k^*) = M_k$

$f(\underline{x}_k^*) = m_k$.

If we pick any other point x_k^* in $[x_{k-1}, x_k]$, we'll have

$$\text{Area}(r_k) = f(\underline{x}_k^*)\Delta x_k \leq f(x_k^*)\Delta x_k \leq f(\overline{x}_k^*)\Delta x_k = \text{Area}(R_k)$$

Call $\underbrace{\sum_{k=1}^{n} f(x_k^*)\,\Delta x_k}_{\text{area of rectangle with}} = $ Riemann Sum

area of rectangle with base $= [x_{k-1}, x_k]$
height $= f(x_k^*)$

We get $\underset{\substack{\| \\ \text{Lower Riemann Sum}}}{A(r)} \leq$ Riemann Sum $\leq \underset{\substack{\| \\ \text{Upper Riemann Sum}}}{A(R)}$

If $\delta = \max\limits_{k=1,\dots n} \Delta x_k \longrightarrow 0$, then <u>all</u> $\Delta x_k \to 0$ & this forces $n \to \infty$.

- If we show $\boxed{\lim\limits_{\delta\to 0} A(r) = \lim\limits_{\delta \to 0} A(R)}$ (*), then by the Squeeze Theorem we'll have $\text{Area}(S) = \text{this limit} = \lim\limits_{\delta\to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$ for $\delta = \max\limits_{k=1,\dots n} \Delta x_k$ & x_k^* any point in $[x_{k-1}, x_k]$

(Special choices: $x_k^* =$ leftmost point $= x_{k-1}$)
$x_k^* =$ rightmost point $= x_k$
$x_k^* =$ midpoint $= \dfrac{x_{k-1}+x_k}{2}$

<u>Definition</u>: We set $\int_a^b f(x)\,dx$ as the area of S, viewed as a limit of Riemann Sums. We call it the definite integral.

<u>Names</u>: $a =$ lower limit of integration , $f(x) =$ integrand
$b =$ upper ——————— $x =$ variable of integration.

<u>Q</u>: How can we garantee (*)? Continuity of f will show that $m_k = f(\underline{x}_k^*)$ & $M_k = f(\overline{x}_k^*)$ will be close enough if Δx_k is sufficiently small (See Appendix A5 for details).

<u>Next Time</u>: Examples of integrable & non-integrable functions.