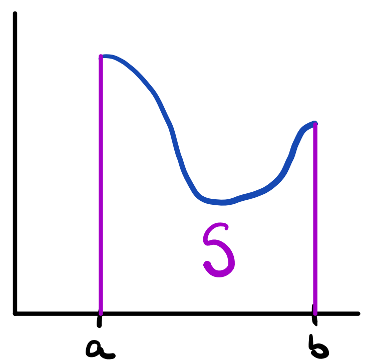
Lecture XXIII: §6.5 The computation of areas as limits
§6.7 Properties of definite integrals

§1 Definite integrals:

Recall: given a continuous function $f:[a,b] \longrightarrow \mathbb{R}_{\geq 0}$ ($\{x \geq 0\}$), we want to find the area of the region S that lies under the curve $y=f(x)$ & above the x-axis



We use 3 approximations to compute Area(S)

1. Overestimate with larger rectangles (entrap S)
   ⇝ Upper Riemann Sums
2. Underestimate with small rectangles (exhaust S)
   ⇝ Lower Riemann Sums
3. Rectangles that are close to the graph
   ⇝ Riemann Sums

All 3 types of rectangles will have:

- base on the x-axis of small length
- height: either a max (for 1), a min (for 2) of f restricted to each base or an arbitrary value $f(x)$ for x in each base (for 3)

STEP 1 Pick points $x_1, \ldots, x_n$ that determine the bases of these ▭

We set $a = x_0$ & $b = x_{n+1}$ for convenience.

We write the differences as
$$\begin{aligned}\Delta x_1 &= x_1 - a\\ \Delta x_2 &= x_2 - x_1\\ &\vdots\\ \Delta x_n &= x_n - x_{n-1}\\ \Delta x_{n+1} &= b - x_n\end{aligned}$$

we assume they are all small (which means n is going to be larger!)

To fix ideas, we can make $\Delta x_k = \dfrac{b-a}{n+1}$ for all $k = 1, 2, \ldots, n+1$.

STEP 2: Use these intervals as the bases of our rectangles.

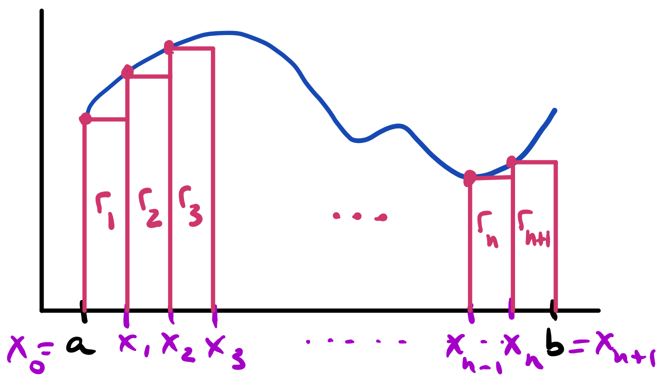
- $n+1$ small rectangles $= r_1, \ldots, r_{n+1}$ & height of $r_k = m_k = \min$ of f on $[x_{k-1}, x_k]$

- $n+1$ large rectangles $= R_1, \ldots, R_{n+1}$ & height of $R_k = M_k = \max$ of f on $[x_{k-1}, x_k]$
- $n+1$ intermediate rectangles $= P_1, \ldots, P_{n+1}$ & height of $P_k = f(x_k^*)$ for some choice of points $x_1^*, \ldots, x_{n+1}^*$ with each x_k^* in $[x_{k-1}, x_k]$

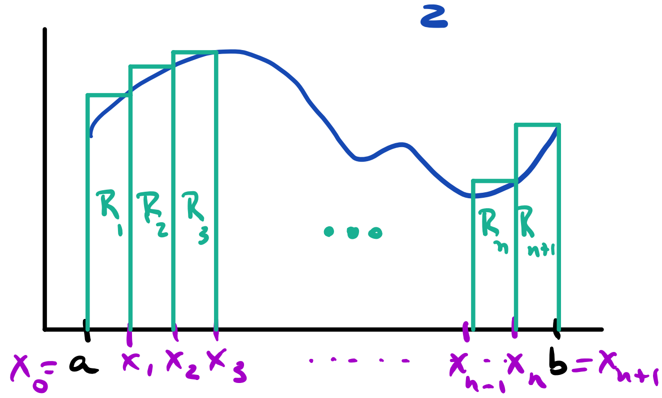
(Popular choices: $x_k^* =$ leftmost point $= x_{k-1}$)

$x_k^* =$ rightmost point $= x_k$

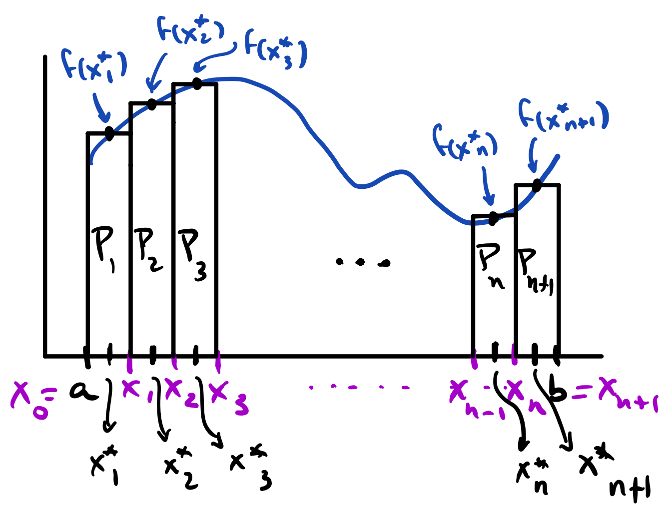
$x_k^* =$ midpoint $= \dfrac{x_{k-1}+x_k}{2}$



LOWER SUMS



UPPER SUMS



Since $\quad m_k = \min\limits_{x_{k-1} \le x \le x_k} (f(x)) \le f(x_k^*) \le M_k = \max\limits_{x_{k-1} \le x \le x_k} (f(x))$

We get $\quad \underbrace{m_k \, \Delta x_k}_{= \text{Area}(r_k)} \le \underbrace{f(x_k^*) \, \Delta x_k}_{= \text{Area}(P_k)} \le \underbrace{M_k \, \Delta x_k}_{= \text{Area}(R_k)}$

STEP 3: Summing up these areas we get 3 Riemann Sums:

$$\underbrace{\sum_{k=1}^{n+1} m_k \, \Delta x_k}_{\text{LOWER Riemann Sum}} \le \underbrace{\boxed{\sum_{k=1}^{n+1} f(x_k^*) \, \Delta x_k}}_{\text{RIEMANN SUM}} \le \underbrace{\sum_{k=1}^{n+1} M_k \, \Delta x_k}_{\text{UPPER Riemann Sum}}$$

Theorem: If f is continuous, then as $\delta = \max_{k=1,\dots,n+1} \Delta x_k \to 0$, we have $n \to \infty$ and

$$\lim_{\delta \to 0} \sum_{k=1}^{n+1} \text{Area}(r_k) = \lim_{\delta \to 0} \sum_{k=1}^{n+1} \text{Area}(R_k)$$

And by "Squeeze Theorem", this limit is also $= \lim_{\delta \to 0} \sum_{k=1}^{n+1} f(x_k^*) \Delta x_k$.

It agrees with Area(S) by the discussion on Lecture XXII.

Definition: $\int_a^b f(x)\,dx = \text{Area}(S) =$ definite integral (Viewed as a limit of Riemann Sums)

Names: a = lower limit of integration, $f(x)$ = integrand

b = upper limit of integration, x = variable of integration.

Definition: We say f is <u>integrable</u> over $[a,b]$ if the limit of the Riemann Sums exists and it is <u>independent</u> of all choices of pts ($x_1, \dots, x_n$ & $x_1^*, \dots, x_{n+1}^*$)

Theorem: Continuous functions are integrable.

⚠ Not every function is integrable (in the sense of Riemann!)

EXAMPLE: $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$ on $[0,1]$

This function is nowhere continuous!

$\rightsquigarrow$ Lower Riemann Sums: $m_k = 0$ for all k so $A(r) = 0$.

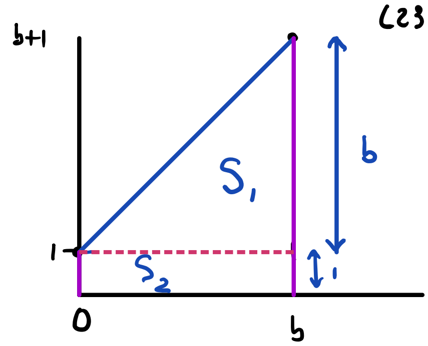
Upper Riemann Sums: $M_k = 0$ for all k so

$$A(R) = \sum_{k=1}^{n+1} 1 \cdot \Delta x_k = \text{length}([0,1]) = 1.$$

So f is not integrable (general Riemann Sums have no limit!)

- This led to the creation of other notions of integrability (by Lebesgue ...) $\rightsquigarrow$ Real Analysis.

§2. Examples:



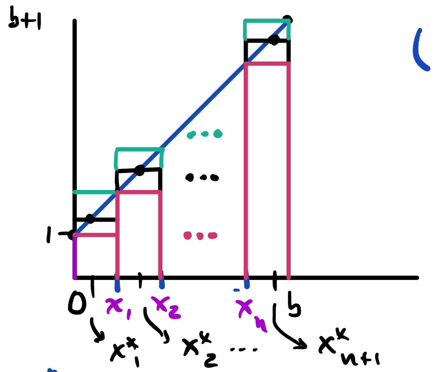
EXAMPLE 1: $f : [0,b] \longrightarrow \mathbb{R}_{\geq 0}$ $\quad f(x) = x+1$

f is continuous, so it's integrable!

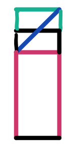
$$\text{Area}(S) = \text{Area}(S_1) + \text{Area}(S_2)$$
$$= \frac{b^2}{2} + 1\cdot b = \frac{b^2+b}{2}$$

- Pick $x_1, \ldots, x_n$ equidistant so $x_k = \frac{b}{n+1}\cdot k$ for $k=1,\ldots,n$

$(x_0 = 0 \quad , x_{n+1} = b.)$



Upper Sums
Lower Sums
General Sums



- Lower Sums = $\underline{x}^*_k = x_{k-1} \rightsquigarrow m_k = f(\underline{x}^*_k) = 1 + x_{k-1}$
- Upper Sums = $\overline{x}^*_k = x_k \rightsquigarrow M_k = f(\overline{x}^*_k) = 1 + x_k$

} because f is increasing

- General Riemann sum $f(x^*_k) = 1 + x^*_k$

$$\text{So } \sum_{k=1}^{n+1} \text{Area}(r_k) = \sum_{k=1}^{n+1} m_k \Delta x_k = \sum_{k=1}^{n+1} (1 + \underbrace{x_{k-1}}_{= \frac{b(k-1)}{n+1}}) \frac{b}{n+1} = \frac{b}{n+1} \sum_{k=1}^{n+1} \left(1 + \frac{b}{n+1}(k-1)\right)$$

$$= \frac{b}{n+1}\left(\sum_{k=1}^{n+1} 1 + \sum_{k=1}^{n+1} \boxed{\frac{b}{n+1}}(k-1)\right) = \frac{b}{n+1}\left((n+1) + \frac{b}{n+1}\sum_{k=1}^{n+1}(k-1)\right)$$

Fixed number!

$$= \frac{b}{n+1}\left(n+1 + \frac{b}{n+1}\left(0+1+2+\cdots+n\right)\right)$$

$$= b + \frac{b^2}{(n+1)^2}\sum_{k=1}^{n} k = b + \frac{b^2}{(n+1)^2}\frac{(n+1)n}{2}$$

Lecture 22 (Prop 1)

$$= b + \frac{b^2}{2}\frac{n}{n+1}$$

$$\text{Similarly: } \sum_{k=1}^{n+1} \text{Area}(R_k) = \sum_{k=1}^{n+1} (1+x_k)\frac{b}{n+1} = \frac{b(n+1)}{n+1} + \sum_{k=1}^{n+1} \frac{b}{n+1} x_k$$

$$= b + \frac{b^2}{n+1}\frac{(n+1)(n+2)}{2} = b + \frac{b^2}{2}\frac{n+2}{n+1}.$$

Having $\delta = \frac{b}{n+1} \longrightarrow 0$ forces $n \to \infty$

So $\lim\limits_{\delta \to 0} \sum\limits_{k=1}^{n+1} \text{Area}(r_k) = \lim\limits_{\delta \to 0} \left(b + \frac{b^2}{2} \frac{n}{n+1} \right) = \lim\limits_{n \to \infty} b + \frac{b^2}{2}\frac{n}{n+1}$

$= b + \frac{b^2}{2}$

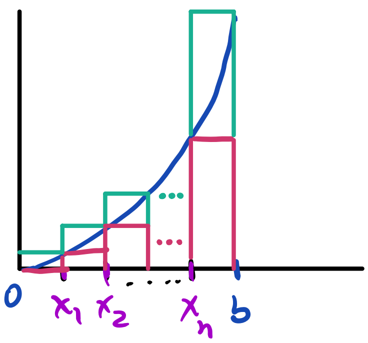
$\lim\limits_{\delta \to 0} \sum\limits_{k=1}^{n+1} \text{Area}(R_k) = \lim\limits_{\delta \to 0} \left(b + \frac{b^2}{2} \frac{n+2}{n+1} \right) = \lim\limits_{n \to \infty} b + \frac{b^2}{2}\frac{n+1}{n+2}$

$= b + \frac{b^2}{2}$

So $\int_0^b (x+1)\,dx = b + \frac{b^2}{2} = \lim\limits_{n \to \infty} \sum\limits_{k=1}^{n} (1 + x_k^*) \frac{b}{n}$

⚠ This calculation doesn't show that the choice of points doesn't matter, but this fact follows from the Theorem (we know f is integrable!)

EXAMPLE 2: $f : [0,b] \longrightarrow \mathbb{R}^2$ $\quad f(x) = x^2$ $\quad$ Area = ? (f is integrable!)



- Pick $x_1, \dots, x_n$ equidistant

  $x_k = \frac{bk}{n+1}$, $\quad 0 = x_0$, $\quad b = x_{n+1}$

- f is increasing so $\underline{x}^*_k = x_{k-1} = \frac{b(k-1)}{n+1}$

  $\overline{x}^*_k = x_k = \frac{bk}{n+1}$

Lower Sums: $\sum\limits_{k=1}^{n+1} \text{Area}(r_k) = \sum\limits_{k=1}^{n+1} (\underline{x}^*_k)^2 \frac{b}{n+1} = \sum\limits_{k=1}^{n+1} \left(\frac{b(k-1)}{n+1} \right)^2 \frac{b}{n+1}$

$= \sum\limits_{k=1}^{n+1} \boxed{\frac{b^3}{(n+1)^3}} (k-1)^2 = \frac{b^3}{(n+1)^3} \sum\limits_{k=1}^{n+1} (k-1)^2 = \frac{b^3}{(n+1)^3} \left(0^2 + 1^2 + 2^2 + \dots + n^2 \right)$

↖ fixed number

$= \frac{b^3}{(n+1)^3} \sum\limits_{k=1}^{n} k^2 = \frac{b^3}{(n+1)^3} \frac{n(n+1)(2n+1)}{6} = \boxed{\frac{b^3}{6} \frac{n}{n+1} \frac{2n+1}{n+1}}$

Lecture 22 - Prop 2

Upper Sums: $\sum\limits_{k=1}^{n+1} \text{Area}(R_k) = \sum\limits_{k=1}^{n+1} (\overline{x}^*_k)^2 \frac{b}{n+1} = \sum\limits_{k=1}^{n+1} \left(\frac{bk}{n+1} \right)^2 \frac{b}{n+1} =$

$= \sum\limits_{k=1}^{n+1} \boxed{\frac{b^3}{(n+1)^3}} k^2 = \frac{b^3}{(n+1)^3} \sum\limits_{k=1}^{n+1} k^2 = \frac{b^3}{(n+1)^3} \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}$

fixed number

Lecture 22 - Prop 2

$= \boxed{\frac{b^3}{6} \frac{n+2}{n+1} \frac{2n+3}{n+1}}$

Both sums have the same limit as $n \to \infty$ $\left(\delta = \frac{b}{n+1} \to 0 \right)$

This limit is $\frac{b^3}{6} \cdot 2 = b^3/3$

Conclusion: $\text{Area}(S) = \int_0^b x^2\,dx = \frac{b^3}{3} = \lim_{n\to\infty} \sum_{k=1}^{n+1} (x_k^*)^2 \frac{b}{n+1}$

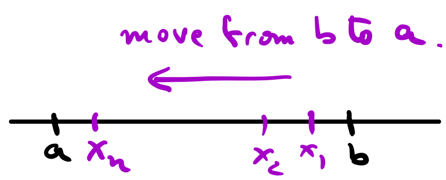
for any choice of $x_1^*, \ldots, x_{n+1}^*$ with $\frac{b(k-1)}{n+1} \leq x_k^* \leq \frac{bk}{n+1}$ for $k=1,\ldots,n+1$

§3 General Properties:

① If $a \leq b$, then $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

(so $\int_a^a f(x)\,dx = 0$)

Reason: Order of $x_1, \ldots, x_n$ is descending!



$\Delta x_k = x_{k+1} - x_k < 0$

↳ opposite sign to the increment going from a to b.

$f(x_k^*)$ is the same for both orders

② Additivity I: If $a < c < b$: $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

③ Scalar Multiplication: $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$

Reason: $(cf)(x_k^*) = c\,f(x_k^*)$ & use Limit Laws.

④ Additivity II: $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

if both f & g are continuous.

Why? Again, by Limit Laws + $(f+g)(x_k^*) = f(x_k^*) + g(x_k^*)$

⚠ Need to work with general Riemann Sums since

Upper R.S. for f + Upper R.S for g $\neq$ Upper R.S for $(f+g)$

Lower ______ + Lower ______ $\neq$ Lower ______