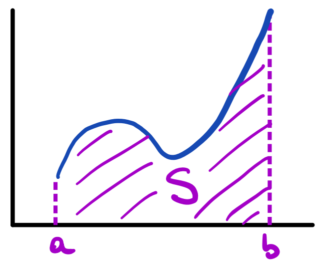
Lecture XXIV: §6.7 Algebraic vs Geometric Areas
§6.6 The Fundamental Theorem of Calculus

§1 Algebraic vs. geometric areas

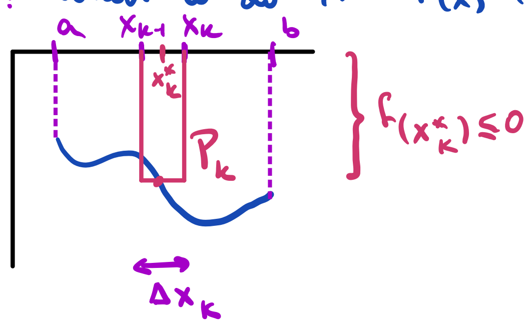


For $f: [a,b] \longrightarrow \mathbb{R}_{\geq 0}$ continuous, we define

- **Geometric Area** = area of the region S enclosed by the x-axis and the graph of f.

$$= \int_a^b \underbrace{f(x)}_{\geq 0}\, dx$$

Q: What to do if $f(x)$ is negative for all x?



For general Riemann Sums:

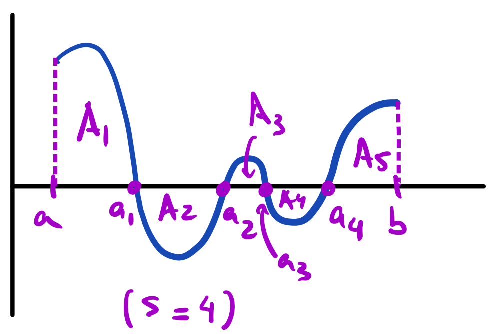
$$f(x^*_k)\, \Delta x_k = -\text{Area}\,(P_k)$$

so
$$\int_a^b f(x)\, dx = -\text{Area}(S) = -\int_a^b (-f(x))\, dx = -\int_a^b |f(x)|\, dx$$

If $f(x)$ is both positive & negative on $[a,b]$, the sign of the integral of f is not predetermined.

Definition:
- $\displaystyle\int_a^b |f(x)|\, dx$ = Geometric Area
- $\displaystyle\int_a^b f(x)\, dx$ = Algebraic (or Signed) Area

Q: Connection to areas under a curve?



STEP 1: Find and order the x-intercepts of f. Call them: $a_1 < \cdots\cdots < a_s$ & write $a = a_0$, $a_{s+1} = b$.

STEP 2: The area bounded by the x-axis and the restriction of f to $[a_{k-1}, a_k]$ gives us $A_k := \int_{a_{k-1}}^{a_k} |f(x)|\,dx$

$$\text{sign}_{A_k}(f) = \begin{cases} + & \text{if } f \geq 0 \text{ on } [a_{k-1}, a_k] \\ - & \text{if } f \leq 0 \text{ \_\_\_\_} \end{cases}$$

$$\text{Geometric Area} = \sum_{k=1}^{5} A_k = \int_a^b |f(x)|\,dx$$

$$\text{Signed Area} = \sum_{k=1}^{5} \text{sign}_{A_k}(f) \cdot A_k = \sum_{k=1}^{5} \text{sign}_{A_k}(f) \int_{a_{k-1}}^{a_k} |f(x)|\,dx$$

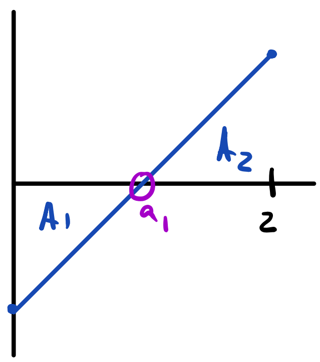
$$= \sum_{k=1}^{5} \int_{a_{k-1}}^{a_k} \underbrace{\text{sign}_{A_k}(f)\,|f(x)|}_{=f(x)}\,dx = \int_a^b f(x)\,dx.$$

(Notice: here we are using a general version of the Additive Property I from Lecture 23)

In our example: Geom Area $= A_1 + A_2 + A_3 + A_4 + A_5$

Signed Area $= A_1 - A_2 + A_3 - A_4 + A_5$.

Example, $f(x) = x - 1$ on $[0,2]$



Zeros of f: only 1 at $a_1 = 1$

$A_1 = \frac{1 \cdot 1}{2} = A_2$

Geometric Area $= A_1 + A_2 = \frac{1}{2} + \frac{1}{2} = 1$

Signed Area $= -A_1 + A_2 = -\frac{1}{2} + \frac{1}{2} = 0$.

Consequence: The properties for definite integrals that we saw in Lecture 23 when $f(x) \geq 0$ are true for any continuous function.

§2. Fundamental Theorem of Calculus:

- Fundamental = it relates differential & integral calculus.
- This result will allow us to compute integrals without using Riemann Sums!

Theorem: Assume $f:[a,b] \longrightarrow \mathbb{R}$ is continuous, and let $F(x)$ be ANY antiderivative of f (recall: we used the notation $F = \int f(x)\,dx$)

Then, the signed area between the graph of f and the x-axis is:

$$\int_a^b f(x)\,dx = F(b) - F(a) =: F(x)\Big|_a^b$$

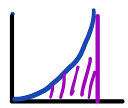
Example (last time) Using Riemann Sums we saw

$$\int_0^b (1+x)\,dx = b + \frac{b^2}{2}$$ & $F(x) = x + \frac{x^2}{2}$ is an antiderivative of $f(x) = 1+x$.

Example above: $f(x) = x-1$ on $[0,2]$

Signed Area $= 0 = \int_0^2 (x-1)\,dx$; $F(x) = \frac{x^2}{2} - x$; $F(0)=0$ & $F(2) = 2-2 = 0$.

Example: $f(x) = x^2$ on $[0,b]$ (left as an exercise last time)

Using Riemann Sums we get Area() $= \frac{b^3}{3} = \int_0^b x^2\,dx$

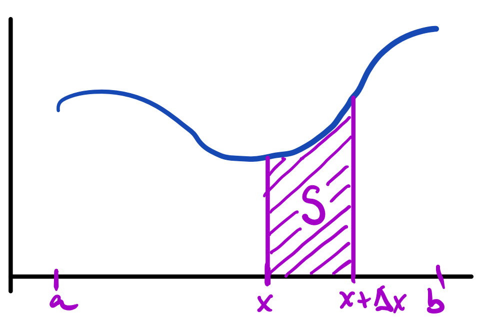
Check: $F(x) = \frac{x^3}{3}$ is an antiderivative of $f(x) = x^2$, $F(0) = 0$, $F(b) = \frac{b^3}{3}$

Proof idea of the Fundamental Theorem of Calculus (Leibniz - Newton)

For simplicity, we assume $f \geq 0$. Otherwise, we work with pieces where the sign is constant & use additivity (For example: if f has only one zero on $[a,b]$, called c, we have

$$\int_a^b f(x) = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$
$$= (\cancel{F(c)} - F(a)) + (F(b) - \cancel{F(c)})$$
$$= F(b) - F(a)$$

In general: All contributions from the zeros of f will cancel out.)



For $f:[a,b] \longrightarrow \mathbb{R}_{\geq 0}$ we define the (signed) area function $A:[a,b] \longrightarrow \mathbb{R}$

$$A(x) = \int_a^x f(t)\,dt$$

Note that the variable of integration is now called t, to avoid confusions.

We expect $A(x)$ to be a nice smooth function when f is continuous.

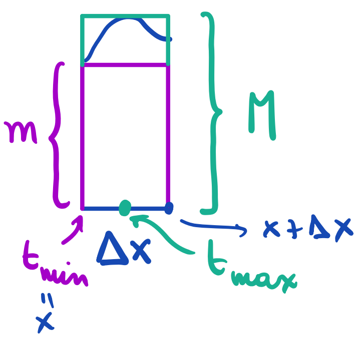
We certify this is true by computing $\frac{dA}{dx}$ via the method of increments

$$\frac{dA}{dx} = \lim_{\Delta x \to 0} \frac{A(x+\Delta x) - A(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\text{Area}(S)}{\Delta x}$$

We want to show: $\frac{dA}{dx} = f(x)$

STEP 1: We want to show that Area(S) is small if Δx is small.

How? Use max & min estimates



$$0 \leq m = \min\{f(t) \text{ for } t \text{ in } [x, x+\Delta x]\}$$

$$0 \leq M = \max\{ \text{———} \}$$

So $\quad m\,\Delta x \leq \text{Area}(S) \leq M\,\Delta x$

$$\boxed{m \leq \frac{\text{Area}(S)}{\Delta x} \leq M} \quad (*)$$

STEP 2: Since f is continuous, the Extreme Value Theorem says $m = f(t_{min})$ & $M = f(t_{max})$ for some t_{min}, t_{max} in $[x, x+\Delta x]$

- If $\Delta x \to 0$, then $t_{min}, t_{max} \to x$ & by the continuity of f

$$m = f(t_{min}) \xrightarrow[\Delta x \to 0]{} f(x)$$

$$M = f(t_{max}) \xrightarrow[\Delta x \to 0]{} f(x)$$

Then, by the Squeeze Theorem applied to $(*)$ we get

$$\lim_{\Delta x \to 0} \frac{\text{Area}(S)}{\Delta x} = f(x)$$

- Alternative argument: Use the Intermediate Value Theorem for f over the interval $[t_{min}, t_{max}]$ (if $t_{min} < t_{max}$) or $[t_{max}, t_{min}]$ (if $t_{min} > t_{max}$) to conclude from $(*)$

that $\frac{\text{Area}(S)}{\Delta x} = f(x^*)$ for some x^* in this interval

Now: x^* is also in $[x, x+\Delta x]$ & f is continuous, so again we see $\lim_{\Delta x \to 0} \frac{\text{Area}(S)}{\Delta x} = \lim_{\Delta x \to 0} f(x^*) = f(x)$ since $x^* \xrightarrow[\Delta x \to 0]{} x$.

STEP 3: By definition $A(x) = \int_a^x f(t)\,dt$ is an antiderivative for $f(x)$. By uniqueness, we can find a constant C with $A(x) = F(x) + C$ for all x in $[a,b]$.

We can find C by evaluating at convenient x's.

$$0 = \int_a^a f(t)\,dt = A(a) = F(a) + C \quad \text{so} \quad C = -F(a)$$

We conclude $A(x) = F(x) - F(a)$ for all x.

Evaluating at $x = b$ gives $A(b) = \int_a^b f(x)\,dx = F(b) - F(a)$.

Q: Why is the choice of antiderivative not important?

A: Any other choice will differ from $F(x)$ by a constant B.

If $G(x) = F(x) + B$, then $G(b) - G(a) = (F(b)+B) - (F(a)+B) = F(b) - F(a)$

Note: The same proof works no matter the sign of f, just replace $\frac{\text{Area}(S)}{\Delta x}$ by $\frac{\text{signed Area}(S)}{\Delta x}$

Consequences: ① If f is continuous $\left(\int_a^x f(t)\,dt\right)' = A(x)' = f(x)$

② $\left(\int_x^a f(t)\,dt\right)' = \left(-\int_a^x f(t)\,dt\right)' = -f(x)$

③ f & u continuous $\frac{d}{dx}\left(\int_a^{u(x)} f(t)\,dt\right) = \frac{d}{dx}\left(A(u(x))\right) = A'_{(u(x))} \cdot u'$

$$= f(u(x)) \cdot u'(x)$$

Ex: $\int^{\sin x} (t+1)\,dt = (\sin x + 1) \cdot \cos x$.