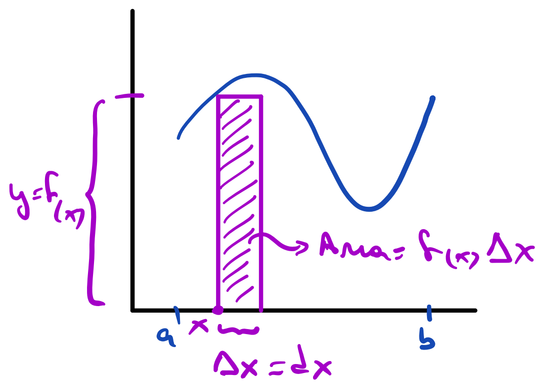
Lecture XXV: §7.1. The intuitive meaning of integration
§7.2. The area between two curves

Recall



Fundamental Theorem of Calculus: If $f:[a,b] \to \mathbb{R}$ is continuous, then $A_{(x)} = \int_a^x f_{(t)}\,dt$ is the signed area of the region between the graph of f restricted to $[a,x]$ & the x-axis. If $F_{(x)}$ is any antiderivative of $f_{(x)}$, we have

$$A(b) = \int_a^b f_{(t)}\,dt = F(b) - F_{(a)} =: F_{(x)}\Big|_a^b .$$

Consequence 1: The proof of FTC showed that $A_{(x)}$ is an antiderivative of $f_{(x)}$, so $\boxed{A'_{(x)} = f_{(x)}.}$

Example: $\left(\int_0^x \sin t \, dt\right)' = \sin x$. Compare with FTC $\int_0^x \sin t\,dt = -\cos t\Big|_0^x = -\cos x + 1$

Consequence 2:

- $\left(\int_x^a f_{(t)}\,dt\right)' = \left(-\int_a^x f_{(t)}\,dt\right)' = -\left(\int_a^x f_{(t)}\,dt\right)' = -f_{(x)}$

- If f is continuous & $u_{(x)}$ is differentiable, then:

$$\frac{d}{dx}\left(\int_a^{u(x)} f_{(t)}\,dt\right) = f_{(u_{(x)})}\cdot u'_{(x)}$$

Why? $(LHS) = \frac{d}{dx} A(u_{(x)}) \underset{\text{Chain Rule}}{=} A'_{(u_{(x)})}\cdot u'_{(x)} \underset{\text{Consequence 1}}{=} f'_{(u_{(x)})}\cdot u'_{(x)}$.

Example: $\int_0^{x^3+\cos x} \sin t\,dt = \sin(x^3+\cos x)\cdot(3x^2 - \sin x)$.

Q What about differentials?

$\underbrace{dA}_{\text{"element of area"}} = A'\,dx = f\,dx$ = signed area of the rectangle with base $[x, x+\Delta x]$

Consequence 3: If f is continuous & $u_{(x)}, v_{(x)}$ are differentiable, then:

$$\frac{d}{dx}\left(\int_{v(x)}^{u(x)} f_{(t)}\,dt\right) = f_{(u_{(x)})}\,u'_{(x)} - f_{(v_{(x)})}\,v'_{(x)}$$

Why? Add an intermediate point!

$$\int_{v(x)}^{u(x)} f(t)\,dt = \int_{v(x)}^{a} f(t)\,dt + \int_{a}^{u(x)} f(t)\,dt = -\int_{a}^{v(x)} f(t)\,dt + \int_{a}^{u(x)} f(t)\,dt$$

So, $$\frac{d}{dx}\left(\int_{v(x)}^{u(x)} f(t)\,dt\right) = \frac{d}{dx}\left(-\int_{a}^{v(x)} f(t)\,dt + \int_{a}^{u(x)} f(t)\,dt\right)$$

$$= -\frac{d}{dx}\left(\int_{a}^{v(x)} f(t)\,dt\right) + \frac{d}{dx}\left(\int_{a}^{u(x)} f(t)\,dt\right)$$

$$= -f(v(x))\cdot v'(x) + f(u(x))\cdot u'(x)$$

Consequence 2

$$= f(u(x))\,u'(x) - f(v(x))\,v'(x)$$

§1 The area between two curves:

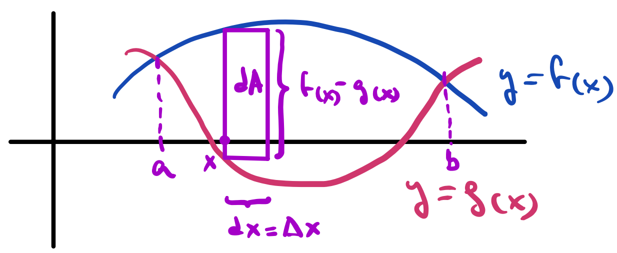
GOAL: Compute the area of a region bounded by 2 smooth curves.

So far, we've done this when one of the curves was the x-axis.

→ **Simplest examples**: f, g satisfy:

(1) $f(a) = g(a)$, $f(b) = g(b)$

(2) For $a < x < b$ $f(x) > g(x)$

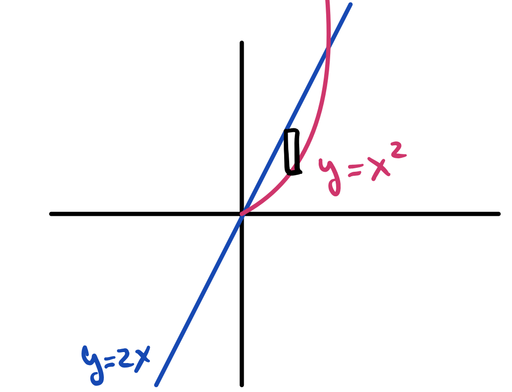


- Height of each strip: $f(x) - g(x) > 0$
- Length of each base $= dx$
- Area of each rectangle $= (f(x) - g(x))\,dx = dA$

So $\text{Area} = \int_a^b (f(x) - g(x))\,dx$.

EXAMPLE 1: $g(x) = x^2$, $f(x) = 2x$. Find the area bounded by these 2 curves:

STEP 1: Draw the curves & find a & b = points where the 2 curves meet.



$$g(x) = f(x)$$
$$x^2 = 2x$$
$$x^2 - 2x = 0$$
$$x(x-2) = 0 \rightsquigarrow \boxed{a=0 \ \& \ b=2}$$

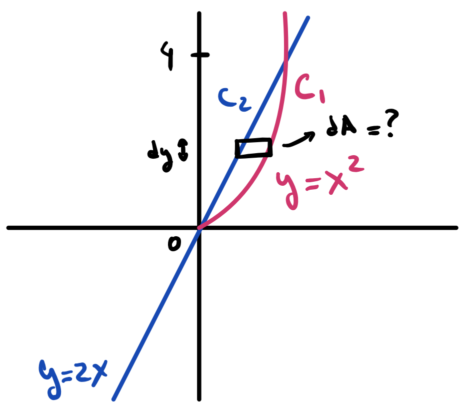
STEP 2: Check which function is larger on $[a,b]$. We do it by picking any point in between.

Pick $\frac{a+b}{2} = 1$ $\quad g(1) = 1^2$ vs $f(1) = 2$ $\rightsquigarrow g(x) < f(x)$ for $0 < x < 2$

STEP 3: Use FTC to find the area $A = \int_a^b dA = \int_a^b (f(x) - g(x))\,dx$

$$A = \int_0^2 (2x - x^2)\,dx = x^2 - \frac{x^3}{3}\Big|_0^2 = \left(4 - \frac{8}{3}\right) - 0 = \boxed{\frac{4}{3}}$$

Q: What happens if we use horizontal strips?



STEP 1: Write the curves as functions of y

C_1: $y = x^2 \rightsquigarrow x = \sqrt{y} = h(y)$

C_2: $y = 2x \rightsquigarrow x = \frac{y}{2} = P(y)$

Bounds: $y = 0$ & 4. & $h(y) > P(y)$ for $0 < y < 4$

STEP 2: Element of area $= dA = (h(y) - P(y))\,dy$

STEP 3: Use FTC

$$A = \int_0^4 h(y) - P(y)\,dy = \int_0^4 \sqrt{y} - \frac{y}{2}\,dy = \frac{2}{3}y^{3/2} - \frac{y^2}{4}\Big|_0^4$$

$$= \left(\frac{2}{3}4^{3/2} - \frac{4^2}{4}\right) - 0 = \frac{2 \cdot 8}{3} - 4 = \boxed{\frac{4}{3}}$$

EXAMPLE 2: Find the area between the curves $x = 3y + y^2$ (parabola) & $x + y + 3 = 0$ (line)

STEP 1: Draw the curves & find a & b = points where the 2 curves meet.

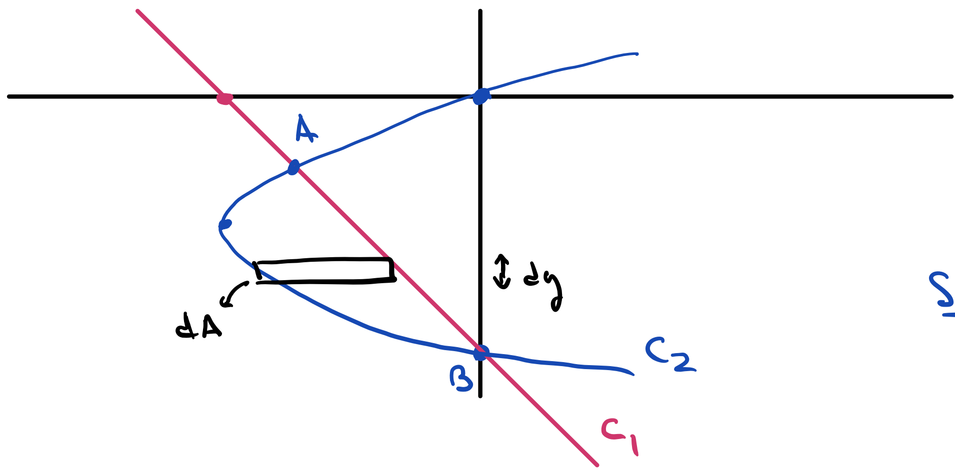
• Vertex of the parabola $\frac{dx}{dy} = 0 = 3 + 2y$ so $y = \frac{-3}{2}$

$x = 3\left(-\frac{3}{2}\right) + \left(\frac{-3}{2}\right)^2 = -\frac{9}{4}$

• y-intercepts: $x = 0 = 3y + y^2 = y(y+3) \rightsquigarrow y = 0$ & $y = -3$

• Line: x-intercepts : $y = 0$: $x + 0 + 3 = 0 \rightsquigarrow x = -3$

y-intercepts $x = 0$: $0 + y + 3 = 0 \rightsquigarrow y = -3$



Intersections:

$3y + y^2 = x = -3 - y$

So $y^2 + 4y + 3 = 0$

Solutions A: $y = -1$ & $x = -2$

B: $y = -3$ & $x = 0$

Seems easier to use horizontal strips: $C_2 : h(y) = 3y + y^2$ in $[-3, -1]$

$C_1 : P(y) = -3 - y$

Compare at $y = -2$ $\quad h(-2) = -6 + 4 = -2$

$p(-2) = -3 + 2 = -1$

so $h_{(x)} < P_{(x)}$ in $[-3, -1]$

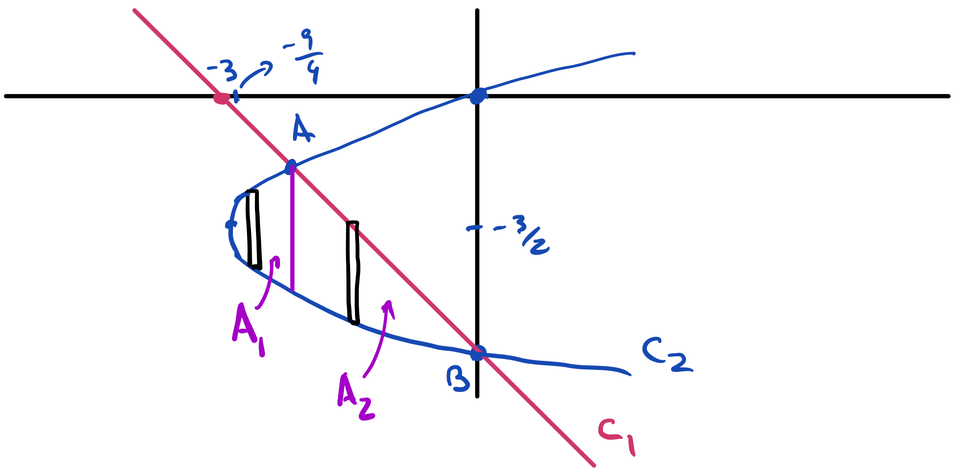
STEP 2: Element of area = $dA = (P(y) - h(y))\,dy$

STEP 3: Use FTC

$$A = \int_{-3}^{-1} P_{(y)} - h_{(y)}\,dy = \int_{-3}^{-1} (-3-y) - (3y + y^2)\,dy = \int_{-3}^{-1} -(y^2 + 4y + 3)\,dy$$

$$= \frac{-y^3}{3} - 2y^2 - 3y \Big|_{-3}^{-1} = \left(+\frac{1}{3} - 2 + 3\right) - \left(\frac{27}{3} - 18 + 9\right) = \boxed{\frac{4}{3}}$$

Q: What if we use vertical strips?



We must divide the region into 2 parts, because the "top" curve is really 2 curves (top part of the side parabola & the line).

Area = $A_1 + A_2$.

STEP 1: Divide the picture by a vertical line through $A = (-2,-1)$

STEP 2: Write the enclosing curves in terms of x.

$$y^2 + 3y - x = 0 \rightsquigarrow y = \frac{-3 \pm \sqrt{9+4x}}{2} = -\frac{3}{2} \pm \sqrt{\frac{9}{4} + x} \quad \text{(2 solutions!)}$$

- Top part of the parabola has function $y = -\frac{3}{2} + \sqrt{x + \frac{9}{4}}$ $\left(x \geq -\frac{9}{4}\right)$
- Bottom ———— $y = -\frac{3}{2} - \sqrt{x + \frac{9}{4}}$ $\left(x \geq -\frac{9}{4}\right)$

STEP 3: Use FTC to find A_1 & A_2

$$A_2 = \int_{-2}^{0} \left((-3-x) - \left(-\frac{3}{2} - \sqrt{x+\frac{9}{4}}\right)\right) dx = \int_{-2}^{0} \left(-\frac{3}{2} - x + \sqrt{x+\frac{9}{4}}\right) dx$$

$$= -\frac{3}{2}x - \frac{x^2}{2} + \frac{2}{3}\left(x + \frac{9}{4}\right)^{3/2} \Big|_{-2}^{0}$$

$$= \frac{2}{3}\left(\frac{9}{4}\right)^{3/2} - \frac{2}{3}\left(-2+\frac{9}{4}\right)^{3/2} + \left(0 - \left(-\frac{3}{2}(-2) - \frac{(-2)^2}{2}\right)\right) = \frac{7}{6}$$

$$A_1 = \int_{-9/4}^{-2} \left(\left(-\frac{3}{2} + \sqrt{x+\frac{9}{4}}\right) - \left(-\frac{3}{2} - \sqrt{x+\frac{9}{4}}\right)\right) dx = 2\int_{-9/4}^{-2} \sqrt{x+\frac{9}{4}}\, dx$$

$$= 2\,\frac{2}{3}\left(x+\frac{9}{4}\right)^{3/2} \Big|_{-9/4}^{-2} = \frac{4}{3}\left(\left(-2+\frac{9}{4}\right)^{3/2} - 0\right) = \frac{4}{3}\left(\frac{1}{4}\right)^{3/2} = \frac{4}{24} = \frac{1}{6}$$

So Area $= A_1 + A_2 = \frac{7}{6} + \frac{1}{6} = \frac{8}{6} = \boxed{\frac{4}{3}}$ (same as before!)