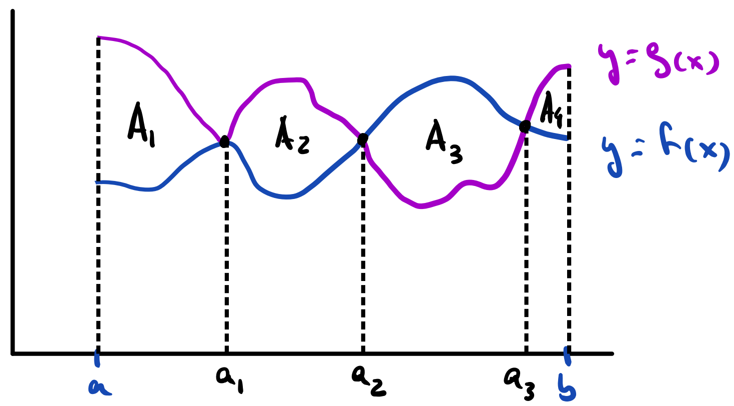
Lecture XXVI: §7.2 The area between two curves
§7.3 Volumes: The Disk Method

§1 The area between two curves with multiple crossings.



Input: Two continuous functions
$f:[a,b] \longrightarrow \mathbb{R}$
$g:[a,b] \longrightarrow \mathbb{R}$

GOAL: Compute the region in between these 2 curves

- Geometric Area $= \int_a^b |g(x)-f(x)|\,dx = A_1 + A_2 + A_3 + A_4$
- Signed Area $= \int_a^b (f(x)-g(x))\,dx = -A_1 - A_2 + A_3 - A_4$

Q: How to compute A_1, A_2, A_3, A_4?

STEP 1: Find all crossings of the 2 curves between a & b:
$(f(x) = g(x)$ with $a \leq x \leq b) \rightsquigarrow a_1, a_2, a_3$ in the picture.

STEP 2:

$A_1 = \int_a^{a_1} g(x) - f(x)\,dx \qquad (g(x) \geq f(x) \text{ in } [a,a_1])$

$A_2 = \int_{a_1}^{a_2} g(x) - f(x)\,dx \qquad (g(x) \geq f(x) \text{ in } [a_1,a_2])$

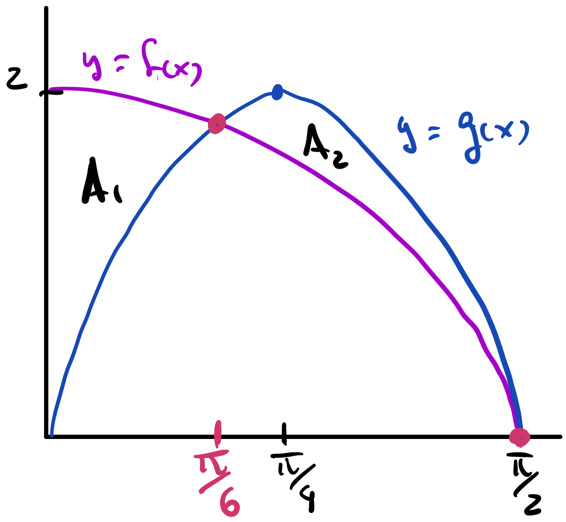
$A_3 = \int_{a_2}^{a_3} f(x) - g(x)\,dx \qquad (g(x) \leq f(x) \text{ in } [a_2,a_3])$

$A_4 = \int_{a_3}^{b} g(x) - f(x)\,dx \qquad (g(x) \geq f(x) \text{ in } [a_3,b])$

Need to compare g & f in between crossings to decide if
$|g(x)-f(x)| = g(x)-f(x)$ or $f(x)-g(x)$.

- We compute the integrals via FTC.

Example: Find the area between $y = 2\cos x$ & $y = 2\sin 2x$ on $[0, \pi/2]$



- $f(x) = 2\cos x$ & $g(x) = 2\sin 2x$
- Compute intersections:

$f(x) = g(x)$

$2\cos x = 2\sin 2x$

$\cos x = \sin 2x = 2\cos x \sin x$

$\cos x(1 - 2\sin x) = 0$

$\rightsquigarrow \cos x = 0$ or $\sin x = \frac{1}{2}$ $\rightsquigarrow$ 2 pts $a_1 = \frac{\pi}{2}$, $a_2 = \frac{\pi}{6}$

$\cos x = 0$, x in $[0, \pi/2]$: $x = \frac{\pi}{2}$

$\sin x = \frac{1}{2}$, x in $[0, \pi/2]$: $x = \frac{\pi}{6}$

- Area $= A_1 + A_2 = \frac{1}{2} + \frac{1}{2} = \boxed{1}$ $\underline{\underline{\text{vs}}}$ • Signed Area $= A_1 - A_2 = \boxed{0}$

- Compare 2 functions:
  - $f(0) = 2$ vs $g(0) = 0$ $\rightsquigarrow f(x) \geq g(x)$ on $[0, \frac{\pi}{6}]$
  - $f(\frac{\pi}{4}) = 2\frac{\sqrt{2}}{2} = \sqrt{2}$ vs $g(\frac{\pi}{4}) = 2$ $\rightsquigarrow f(x) \leq g(x)$ on $[\frac{\pi}{4}, \frac{\pi}{2}]$

$$A_1 = \int_0^{\pi/6} f(x) - g(x)\, dx = \int_0^{\pi/6} (2\cos x - 2\sin 2x)\, dx =$$

$$= 2\sin x \Big|_0^{\pi/6} - \int_0^{\pi/3} \sin u\, du = 2\,\frac{1}{2} + \cos u \Big|_0^{\pi/3} = 1 + \frac{1}{2} - 1 = \frac{1}{2}$$

$\begin{cases} u = 2x \\ du = 2\,dx \end{cases}$ $x = 0 \rightsquigarrow u = 0$, $x = \frac{\pi}{6} \rightsquigarrow u = \frac{\pi}{3}$

$$A_2 = \int_{\pi/6}^{\pi/2} g(x) - f(x)\, dx = \int_{\pi/6}^{\pi/2} (2\sin 2x - 2\cos x)\, dx = \int_{\pi/3}^{\pi} \sin u\, du - 2\sin x \Big|_{\pi/6}^{\pi/2}$$

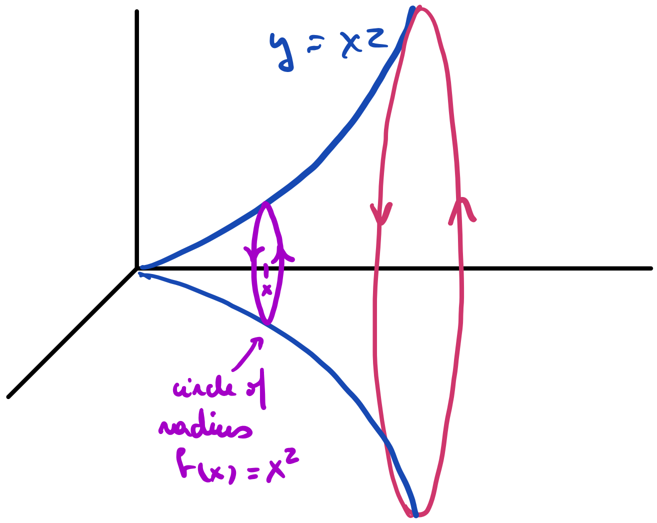
$u = 2x$, $du = 2\,dx$; $x = \frac{\pi}{6} \rightsquigarrow u = \frac{\pi}{3}$, $x = \frac{\pi}{2} \rightsquigarrow u = \pi$

$$= \left(-\cos u \Big|_{\pi/3}^{\pi} - 2\left(1 - \frac{1}{2}\right) = -\cos\pi - \left(-\cos\frac{\pi}{3}\right) - 1 = -(-1) + \frac{1}{2} - 1 = \frac{1}{2}\right.$$

§2 Solid of revolution

- **INPUT:** A positive & continuous function f on $[a,b]$ (Ex $f(x)=x^2$)
- We rotate the graph about the x-axis to get a solid of revolution R

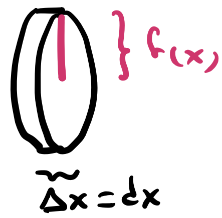
Q What is the volume of R?



Cross section at x: disk of radius $f(x)$

Area of the section $= \pi\,(f(x))^2$

Element of volume: Area of cross section $\cdot\, dx$



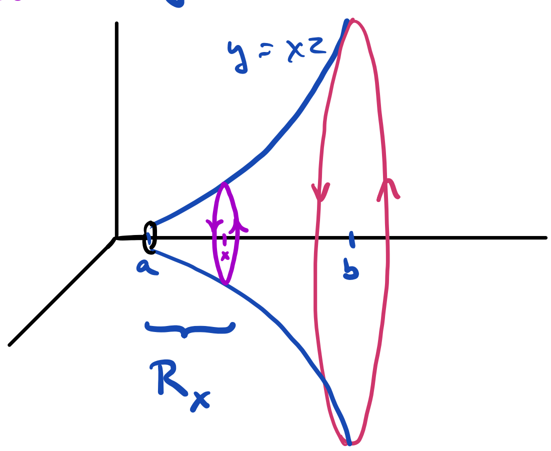
$$\boxed{dV = \pi\,(f(x))^2\,dx}$$

We compute $\mathrm{Vol}(R)$ by covering R with these vertical slices (cross sections) & letting $dx \to 0$. The Riemann Sums give:

$$\boxed{\mathrm{Vol}(R) = \int_a^b dV = \int_a^b \pi\,(f(x))^2\,dx} \quad (*)$$

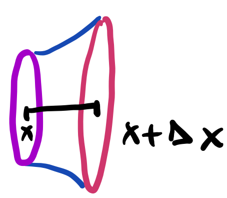
EXAMPLE: $f(x) = x^2$ on $[0,1] \rightsquigarrow \mathrm{Vol}(R) = \int_0^1 \pi\, x^4\,dx = \pi \frac{x^5}{5}\Big|_0^1 = \frac{\pi}{5}$

Q: Why does $(*)$ work?



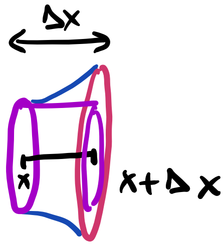
A Define $\mathrm{Vol}(R_x)$ = volume for the solid of revolution between a & x.

So $\mathrm{Vol}(R_x)' = \lim\limits_{\Delta x \to 0} \dfrac{\mathrm{Vol}(R_{x+\Delta x}) - \mathrm{Vol}(R_x)}{\Delta x}$

$P =$ 

$= \lim\limits_{\Delta x \to 0} \dfrac{\mathrm{Vol}(P)}{\Delta x}$

We under- and overestimate Vol(P) by 2 cylinders

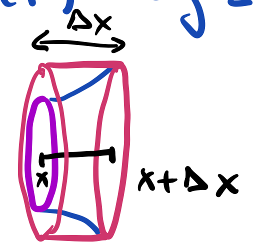


Area $\bigcirc = \pi m^2$

$Vol = \pi m^2 \Delta x$

UNDERESTIMATE

$$m = \min_{x \le c \le x+\Delta x} f(c)$$



Area $\bigcirc = \pi M^2$

$Vol = \pi M^2 \Delta x$

OVERESTIMATE

$$M = \max_{x \le c \le x+\Delta x} f(c)$$

So $\quad$ Vol (inner cylinder) $\le$ Vol (P) $\le$ Vol (outer cylinder)

$$(\pi m^2)\Delta x \le Vol(P) \le (\pi M^2)\Delta x.$$

$$\pi m^2 \le \frac{Vol(P)}{\Delta x} \le M^2$$

Since f is continuous $\pi m^2, \pi M^2 \xrightarrow[\Delta x \to 0]{} \pi f(x)^2$

So the Squeeze Theorem ensures $\displaystyle\lim_{\Delta x \to 0} \frac{Vol(P)}{\Delta x} = \pi f(x)^2$

Conclusion: $\quad Vol(R_x)' = \pi f(x)^2$

Then, by FTC: $\quad Vol(R_x) = \displaystyle\int_a^x \pi f(t)^2\, dt + C$

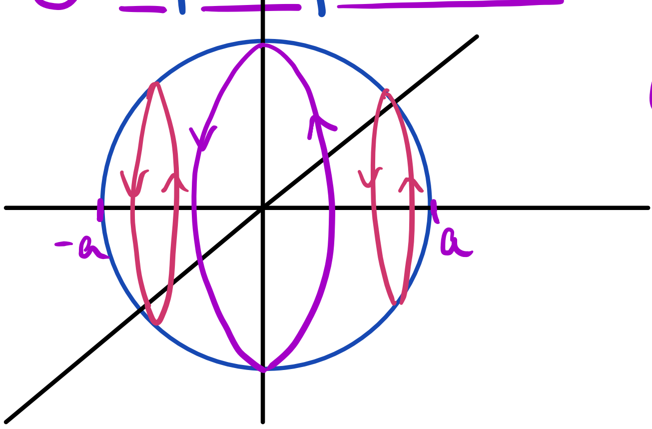
We determine C by evaluating at $x=a$.

$Vol(R_a) = 0 = 0 + C \quad$ so $\quad C = 0$.

Conclude: $\quad \boxed{Vol(R) = Vol(R_b) = \int_a^b \pi f(t)^2\, dt}$

Remark: We can use this idea to recover volumes of some familiar solids of revolution (spheres, cones, cylinders)

① Sphere of radius a : Rotate a half-circle of radius a



Q: What's $f(x)$?

$x^2+y^2=a^2$

so $y=\sqrt{a^2-x^2}$

$$f(x)=\sqrt{a^2-x^2}$$

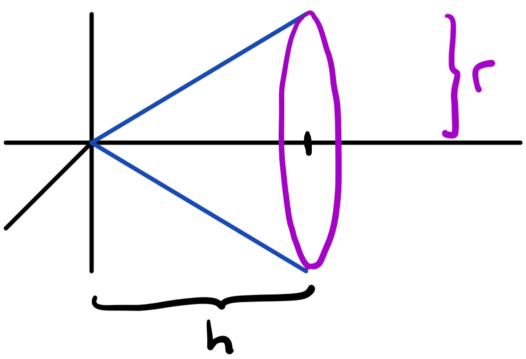
Endpoints : $-a$ & a

$$dV=\pi\,(f(x))^2\,dx=\pi\,(a^2-x^2)\,dx$$

$$\mathrm{Vol}(\text{sphere})=\int_{-a}^{a}\pi\,(a^2-x^2)\,dx=\pi\left(a^2x-\frac{x^3}{3}\right)\Big|_{-a}^{a}$$

$$=\pi\left(a^3-\frac{a^3}{3}-\left(-a^3+\frac{a^3}{3}\right)\right)=\boxed{\frac{4\pi a^3}{3}}$$

② Cone of height $=h$ & radius of base $=r$: Rotate a segment



• Function $y=mx+b$ with $y(0)=0$, $y(h)=r$

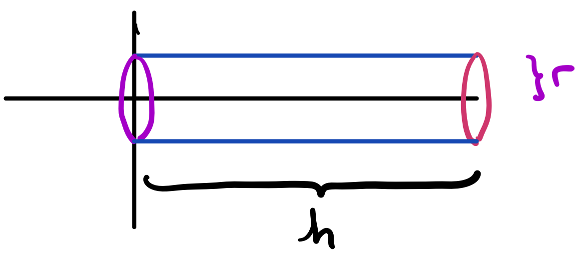
$$\leadsto\ y=f(x)=\frac{r}{h}x$$

• Endpoints : 0 & h

$$dV=\pi\,f(x)^2\,dx=\pi\,\frac{r^2}{h^2}x^2\,dx$$

$$\mathrm{Vol}(\text{Cone})=\int_0^h\pi\,\frac{r^2}{h^2}x^2\,dx=\pi\frac{r^2}{h^2}\frac{x^3}{3}\Big|_0^h=\frac{\pi r^2h^3}{3h^2}=\boxed{\frac{\pi r^2h}{3}}$$

③ Cylinder of radius r & height h : rotate a horizontal segment



Function $f(x)=r$

Endpoints : 0 & h

$$dV=\pi\,r^2\,dx\ \leadsto\ \mathrm{Vol}=\int_0^h\pi r^2\,dx=\boxed{\pi r^2h}$$

⚠ We used it to prove (*).