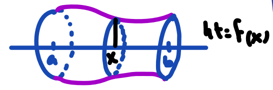
Lecture XXVII: §7.3 Volumes via moving slices & the Washer method
§7.4: Volumes: The Method of cylindrical shells

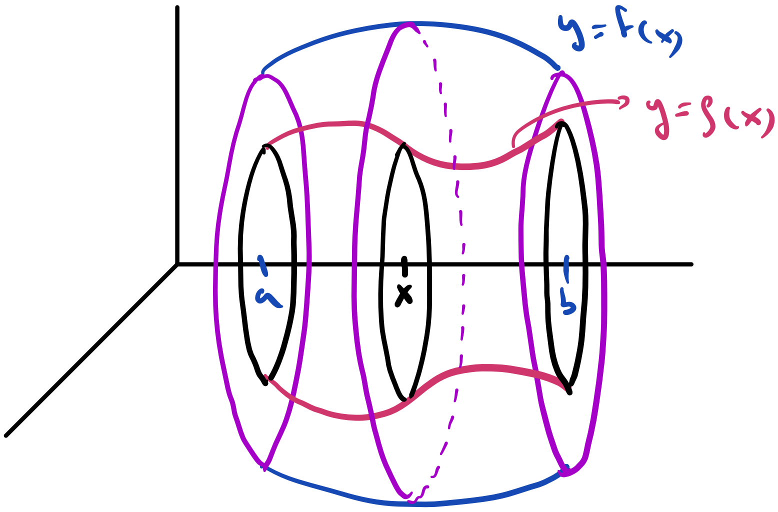
Last Time: We described solids of revolution & computed their volumes via the disk method



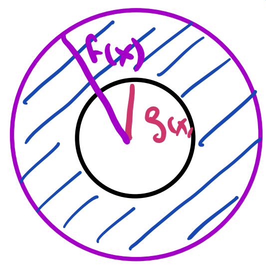
$V = \int_a^b \pi (f(x))^2 \, dx$

§1 Washer Method:

- **INPUT**: 2 positive & continuous functions f, g on $[a,b]$ with $g(x) \leq f(x)$
- Rotate both graphs & get a surface of revolution R



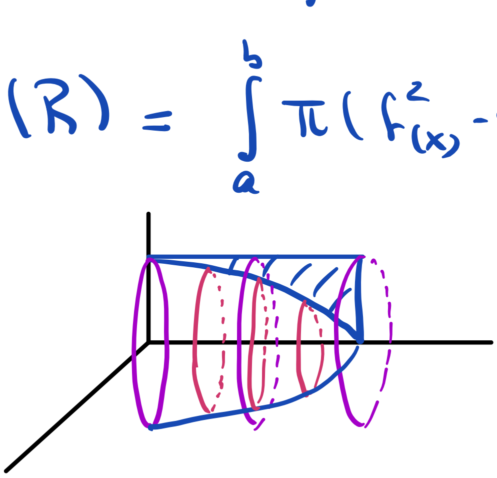
- Cross Section = washer = difference of 2 disks of radii $f(x)$ & $g(x)$



$$dV = \text{Area(Washer)} \cdot dx = \pi \left(f^2_{(x)} - g^2_{(x)} \right) dx$$

$$\rightsquigarrow \quad \text{Vol}(R) = \int_a^b \pi \left(f^2_{(x)} - g^2_{(x)} \right) dx$$

Example



$f(x) = 1 \qquad g(x) = \sqrt{\cos x}$

with $0 \leq x \leq \frac{\pi}{2}$

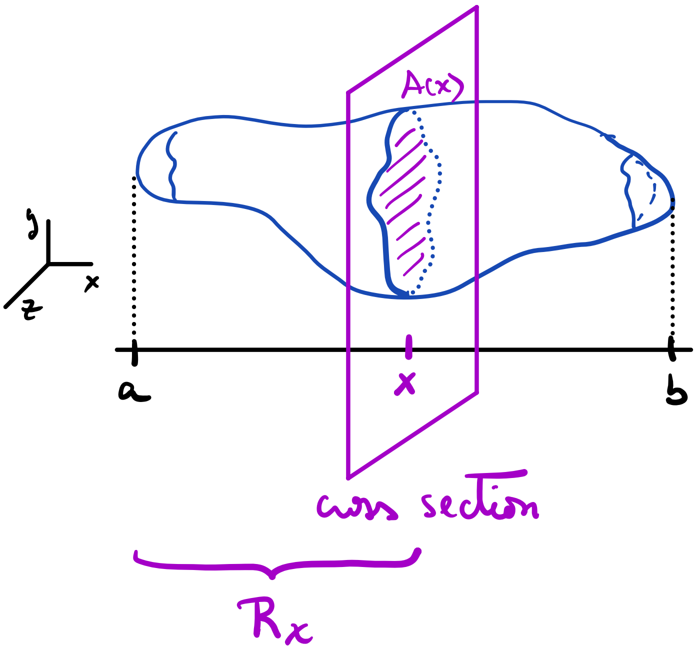
$dV = \pi (1 - \cos x) \, dx$

$$\text{Vol} = \int_0^{\pi/2} \pi (1 - \cos x) \, dx = \pi (x - \sin x) \Big|_0^{\pi/2}$$

$$= \pi \left(\frac{\pi}{2} - 1 \right) - \pi (0 - 0) = \frac{\pi(\pi - 2)}{2}$$

§2. Volumes via moving slices

Next: we want to mimic the volume computation techniques for solids of revolution for more general solids. We will compute volumes by analyzing the areas of convenient slices (either vertical or horizontal)



x-slices = x-cross sections have an area, called $A(x)$

- $dV = A(x)\,dx$
  (think of a cylinder with width dx and base the cross section)
- $\mathrm{Vol} = \int_a^b dV = \int_a^b A(x)\,dx$.

Q Why is this formula valid?

The idea is the same as for solids of revolution.

We define R_x = solid between a & x

So $\frac{d}{dx}\mathrm{Vol}(R_x) = \lim_{\Delta x \to 0} \frac{\mathrm{Vol}(R_{x+\Delta x}) - \mathrm{Vol}(R_x)}{\Delta x}$

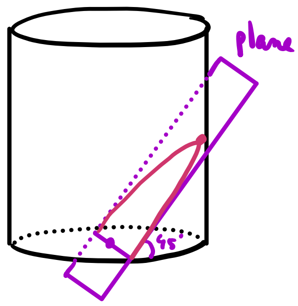
We use continuity of $A(x)$ to show this limit is $A(x)$

(the region between $R_{x+\Delta x}$ & R_x can be under & overestimated with cylinders of width Δx & Volume $= \left(\max_{x<t<x+\Delta x} A(t)\right)\cdot \Delta x$

or $= \left(\min_{x<t<x+\Delta x} A(t)\right)\Delta x$.)

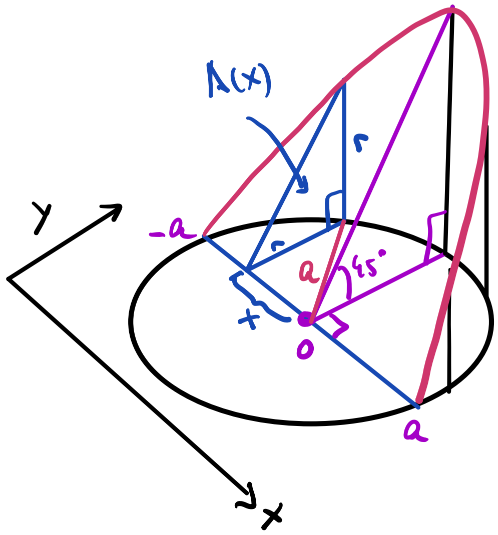
⚠ Hardest part is the determine if x- or y-slices are better for computing.

PROBLEM:



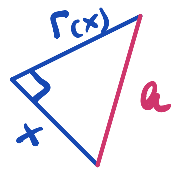
Pick a cylinder with base a circle of radius a & slice it with a plane through the center of the base at a 45° angle. Compute the volume between the plane & the base.

Solution: We look at the cross-sections in the x-direction



Cross-sections = right-isosceles Δ

Side length $r(x)$, & we use



To compute $r(x)$

$$r(x)^2 + x^2 = a^2$$

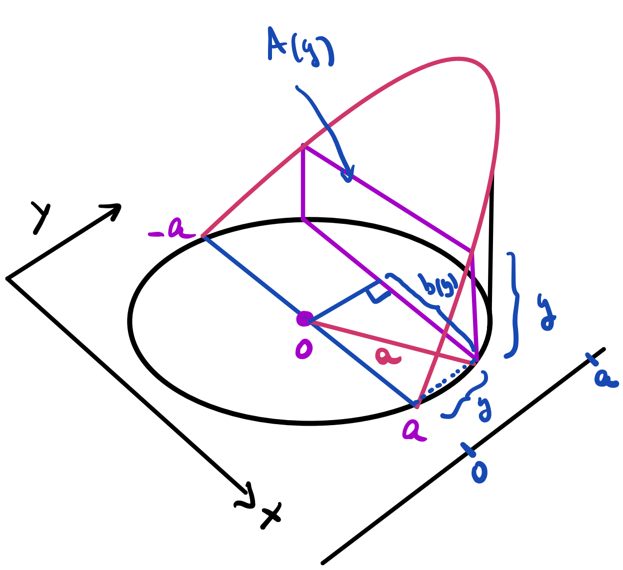
So $A(x) = \dfrac{r(x)\cdot r(x)}{2} = \dfrac{r(x)^2}{2}$

$$= \frac{a^2-x^2}{2}$$

$$\text{Volume} = \int_{-a}^{a} A(x)\,dx = \int_{-a}^{a} \left(\frac{a^2-x^2}{2}\right) dx$$

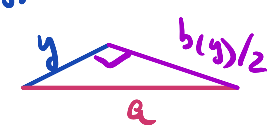
$$= \frac{1}{2}\left(a^2x - \frac{x^3}{3} \Big|_{-a}^{a}\right) = \frac{1}{2}\left(\left(a^3 - \frac{a^3}{3}\right) - \left(-a^3 + \frac{a^3}{3}\right)\right) = \boxed{\frac{2}{3}a^3}.$$

Q What if we use y-slices?



y-slices = rectangles

$\begin{cases} \text{height} = y & (\text{plane is inclined } 45°) \\ \text{base} = b(y) \end{cases}$

We use  to determine $b(y)$

So $a^2 = y^2 + \left(\dfrac{b(y)}{2}\right)^2$ gives

$$b(y) = \sqrt{4(a^2-y^2)} = 2\sqrt{a^2-y^2}$$

Next $\text{Vol} = \int_0^a A(y)\,dy = \int_0^a 2\sqrt{a^2-y^2}\cdot y\,dy$

$$= \int_{a^2}^{0} 2\sqrt{u}\left(-\frac{du}{2}\right) = \int_0^{a^2} \sqrt{u}\,du = \frac{2}{3}u^{3/2}\Big|_0^{a^2} = \boxed{\frac{2}{3}a^3}$$

$u = a^2 - y^2$ $\quad$ $y=0 \rightsquigarrow u=a^2$

$du = -2y\,dy$ $\quad$ $y=a \rightsquigarrow u=0$

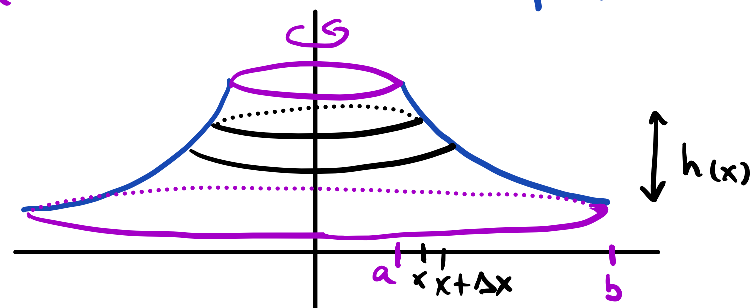
flip the limits of integration

As expected, we get the same value!

§3. Cylindrical Shells:

- INPUT: A decreasing & continuous function $f:[a,b]\longrightarrow\mathbb{R}$ with $a>0$
- Next, we rotate the graph about the y-axis to get a solid of revolution R

Q: What is the volume of R?



$f(b) = m = \min_{a\le c\le b} f(c)$

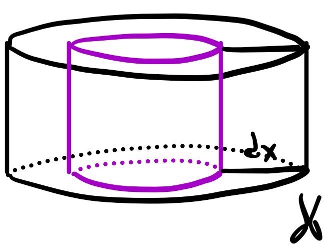
$f(a) = M = \max_{a\le c\le b} f(c)$

Method: Overestimate Vol(R) by covering R with cylindrical shells

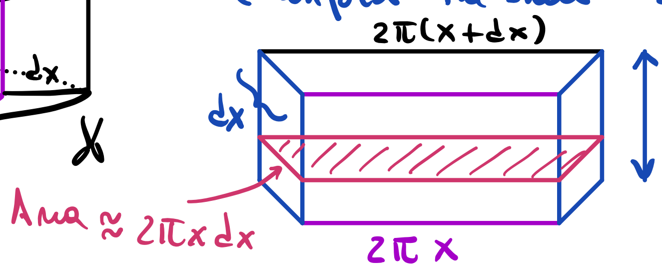
Cylindrical shell = Cylinder 1 − Cylinder 2

Cylinder 1 = cylinder of height $h(x)$ & radius $x+\Delta x = x + dx$

Cylinder 2 = ______________________________ x



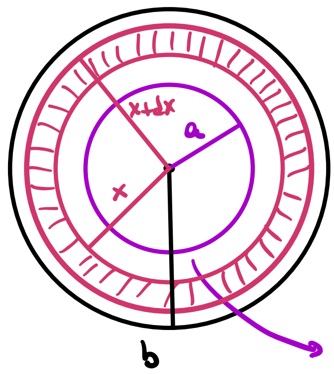
We unfold the shell & get



$h(x) = f(x) - f(b)$

So $dV = 2\pi x\, h(x)\, dx \rightsquigarrow \text{Vol}(R) = \int_a^b dV = \int_a^b 2\pi x\,(f(x) - m)\,dx$

Q What do the cross-sections of these shells look like?



They look like concentric circles of radii x & $x+dx$

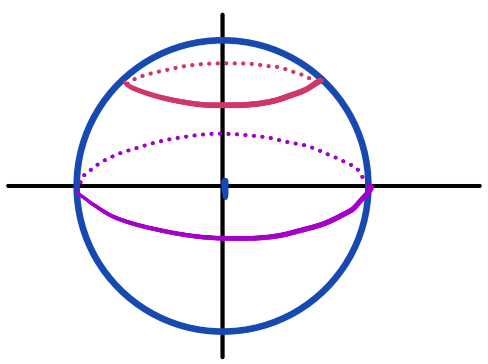
Outer circumference $= 2\pi(x+dx)$

Inner ______ $= 2\pi x$

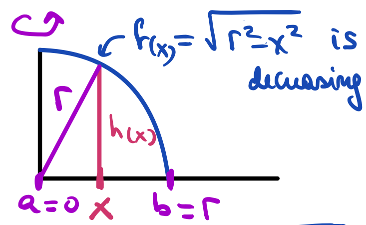
§4 Examples:

① Sphere of radius r

To use cylindrical shells, we compute the volume of the northern hemisphere



. The top half comes from



. Use $h(x)^2 + x^2 = r^2$ to get $h(x) = \sqrt{r^2-x^2}$

. $dV = 2\pi x\, h(x)\, dx = 2\pi x \sqrt{r^2-x^2}\, dx$

So $\mathrm{Vol}(R) = 2\int_0^r 2\pi x\sqrt{r^2-x^2}\, dx = 4\pi\int_0^r x\sqrt{r^2-x^2}\, dx$

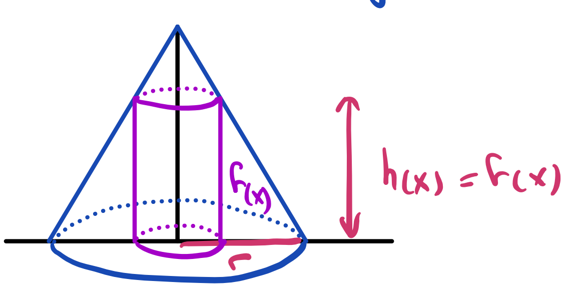
$= 4\pi\int_{r^2}^{0} \sqrt{u}\,\frac{du}{-2} = 2\pi\int_0^{r^2} u^{1/2}\, du$

$\begin{cases} u = r^2-x^2 \\ du = -2x\, dx \end{cases}$ $\quad x=0 \rightsquigarrow u=r^2$, $\quad x=r \rightsquigarrow u=0$

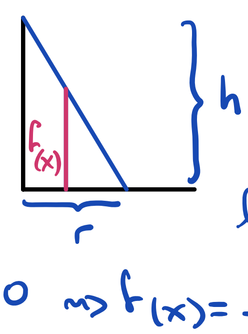
$= 2\pi\frac{2}{3}u^{3/2}\Big|_0^{r^2} = \boxed{\frac{4\pi r^3}{3}}$

(as expected!)

② Cone of height h & radius r



Function:



$f: [0,r] \longrightarrow \mathbb{R}$ linear with $f(0)=h$

& $f(r) = 0 \rightsquigarrow f(x) = \frac{h}{r}(r-x) = -\frac{h}{r}x + h$

we confirm that f is decreasing ✓

$$dV = 2\pi x \, f_{(x)} \, dx = 2\pi x \left(-\frac{h}{r}x + h\right) dx$$

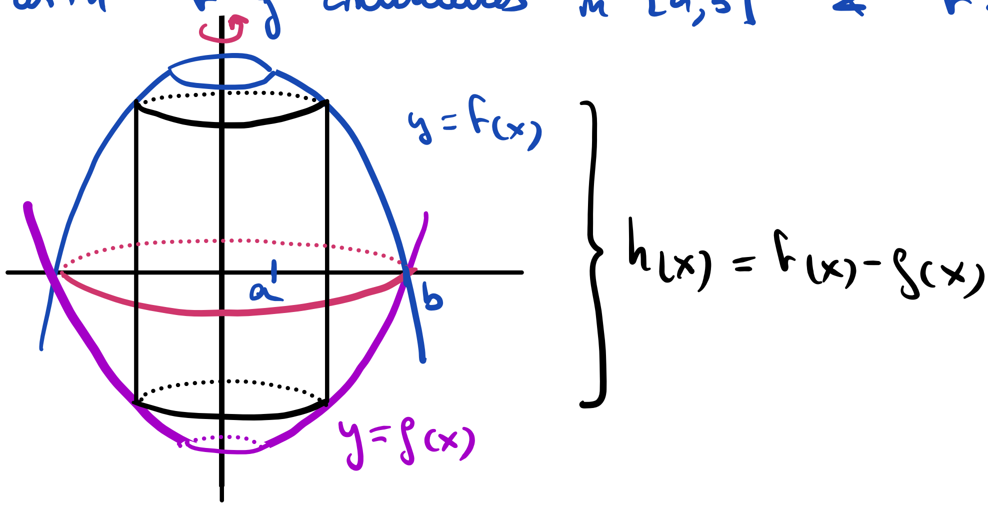
So Vol $= \int_0^r 2\pi x \left(-\frac{h}{r}x + h\right) dx = \int_0^r \left(-2\pi \frac{h}{r} x^2 + 2\pi h x\right) dx$

$$= -2\pi \frac{h}{r} \int_0^r x^2 \, dx + 2\pi h \int_0^r x \, dx$$

$$= -2\pi \frac{h}{r} \frac{x^3}{3} \Big|_0^r + 2\pi h \frac{x^2}{2} \Big|_0^r = -\frac{2}{3} \pi \frac{h}{r} r^3 + \pi h r^2$$

$$= -\frac{2}{3} \pi h r^2 + \pi h r^2 = \boxed{\frac{\pi h r^2}{3}}$$

Q: What if we have 2 bounding curves $y = f_{(x)}$ & $y = g_{(x)}$ with f, g continuous on $[a,b]$ & $f \geq g$



So $dV = 2\pi x \, h_{(x)} \, dx$ by cylinder shells

$$\text{Vol}(R) = \int_a^b 2\pi x \left(f_{(x)} - g_{(x)}\right) dx$$

[Typically, $a = 0$]

Alternative: Use disk method on top part & bottom part separately in the y-direction (⚠ we need to express x in terms of y, which may not be possible, it depends on way $f_{(x)}$ & $g_{(x)}$ are)