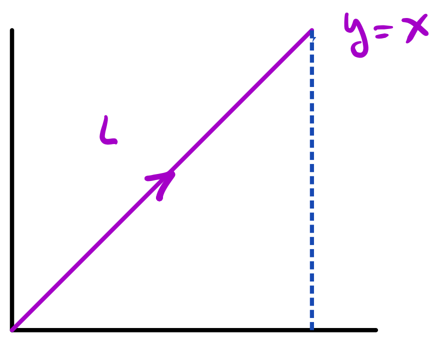
Lecture XXVIII: §7.5 Arc Length
§7.6 The Area of a surface of revolution

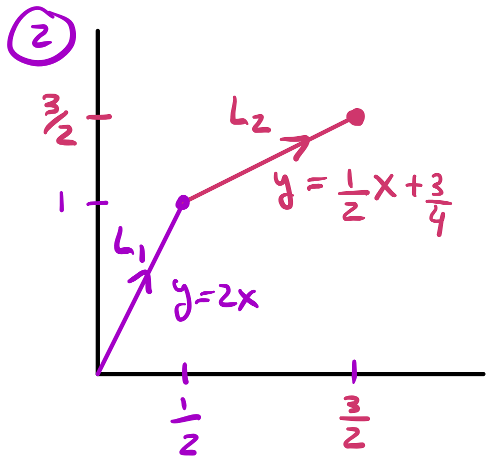
§1. Arc Length

GOAL: Given a curve in the plane, we want to compute its length (= length of a string placed on top of it)

Examples:

① 

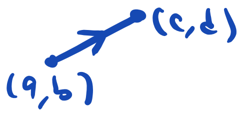
$L=\sqrt{2}$

② 

$L=L_1+L_2=\sqrt{5}$

$L_1=\sqrt{1+\frac{1}{4}}=\frac{\sqrt{5}}{2}$

$L_2=\sqrt{1+\frac{1}{4}}=\frac{\sqrt{5}}{2}$

Conclusion: Length of a polygonal curve (concatenation of segments) is the sum of the lengths of all chords. The length of a chord is computed via Pythagoras' Theorem: 

has length $\sqrt{(c-a)^2+(d-b)^2}$

Q What about general curves?

A Approximate them by polygonals with n points well-distributed, compute the length of the polygonal & take limit when $n\to\infty$.

Theorem: The arc length of the graph $y=f(x)$ on a continuous, differentiable function $f:[a,b]\longrightarrow\mathbb{R}$ with continuous derivative is

$$L=\int_a^b\sqrt{1+(f'(x))^2}\,dx$$

Examples

① $f(x)=y$ so $f'(x)=1$, $a=0$, $b=1$ & $L=\int_0^1\sqrt{1+1}\,dx=\sqrt{2}$

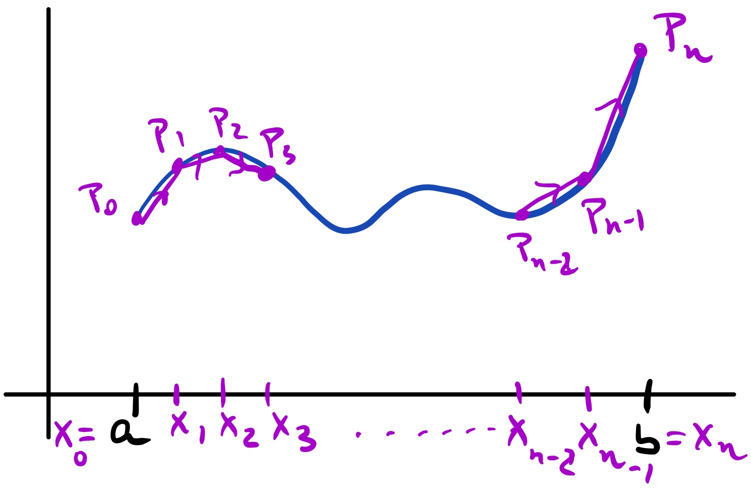
② We have 2 functions

For L_1: $f_1(x) = 2x$ $a=0$, $b=\frac{1}{2}$ $f_1'(x) = 2$ $L_1 = \int_0^{1/2} \sqrt{1+4} = \frac{\sqrt{5}}{2}$

For L_2: $f_2(x) = \frac{1}{2}x + \frac{3}{4}$, $a = \frac{1}{2}$, $b = \frac{3}{2}$ $f_2' = \frac{1}{2}$ $= \int_{1/2}^{3/2} \sqrt{1+\frac{1}{4}} = \frac{\sqrt{5}}{2}$

Q: Why is the formula for L valid?

A Approximate $y=f(x)$ by polygonals with n points well-distributed (obtained by subdividing $[a,b]$, as we did for Riemann Sums), then compute the length of the polygonal & take limit when $n \to \infty$.



$P_0 = (a, f(a))$
$P_1 = (x_1, f(x_1))$
$\vdots$
$P_{n-1} = (x_{n-1}, f(x_{n-1}))$
$P_n = (b, f(b))$

STEP 1: Subdivide $[a,b]$ into n intervals $[a, x_1]$, $[x_1, x_2]$, $\dots$, $[x_{n-2}, x_{n-1}]$, $[x_{n-1}, b]$ of lengths:

$$\begin{cases} \Delta x_1 = x_1 - a \\ \Delta x_2 = x_2 - x_1 \\ \vdots \\ \Delta x_{n-1} = x_{n-1} - x_{n-2} \\ \Delta x_n = b - x_{n-1} \end{cases}$$

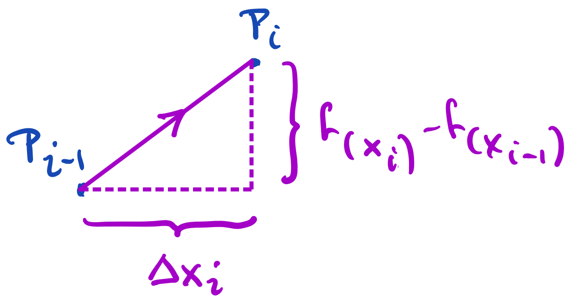
Set $\delta = \max_{1 \le i \le n} \Delta x_i$

(Example: $x_i = \left(\frac{b-a}{n}\right) i$ for all i gives $\Delta x_1 = \Delta x_2 = \dots = \Delta x_n = \frac{b-a}{n}$)

STEP 2: Call $P_0 = (a, f(a))$, $P_1 = (x_1, f(x_1))$, $\dots$, $P_{n-1} = (x_{n-1}, f(x_{n-1}))$, $P_n = (b, f(b))$

Draw the polygonal joining $P_0, P_1, P_2, \dots, P_{n-1}, P_n$.

STEP 3: Compute the length L_i of each $\overline{P_{i-1}P_i}$ $i=1,\dots,n$

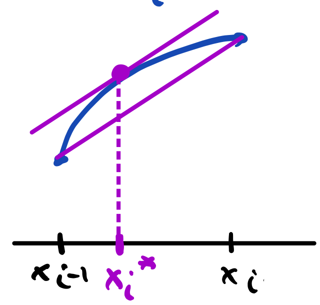


$$L_i = \sqrt{(\Delta x_i)^2 + (f(x_i) - f(x_{i-1}))^2}$$

$$= \Delta x_i + \sqrt{1 + \left(\frac{f(x_{i-1}+\Delta x_i) - f(x_{i-1})}{\Delta x_i}\right)^2}$$

Since f is differentiable on (x_{i-1}, x_i) & continuous on $[x_{i-1}, x_i]$, we can use the Mean Value Theorem to find x_i^* in (x_i, x_{i+1}) with

$$\frac{f(x_{i-1}+\Delta x_i) - f(x_{i-1})}{\Delta x_i} = f'(x_i^*)$$



STEP 4: Compute the length L_{Polyg} of the polygonal:

$$L_{Polyg} = L_1 + \dots + L_n = \sum_{i=1}^{n} \sqrt{1+(f'(x_i^*))^2}\,\Delta x_i$$

← a Riemann Sum!

Since f' is continuous, we know $\sqrt{1+(f'(x))^2}$ is continuous & integrable

Conclusion: $L = \lim\limits_{\max(\Delta_i)\to 0} L_{polyg} = \lim\limits_{\max(\Delta_i)\to 0} \sum_{i=1}^{n} \sqrt{1+(f'(x_i^*))^2}\,\Delta x_i$

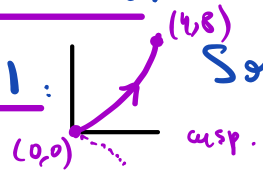
so $$L = \int_a^b \sqrt{1+(f'(x))^2}\,dx$$

⚠ Often, it is very hard the find the antiderivative of $\sqrt{1+(f'(x))^2}$

If so, numerical approximations (with Riemann Sums) are the practical way to "compute" L with enough precision.

EXAMPLE 1: Find the length of the curve $y^2 = x^3$ between $(0,0)$ & $(4,8)$

Soln 1:



Solve for y $y = x^{3/2}$ ⇝ $y' = \frac{3}{2}x^{1/2}$ continuous

So $L = \int_0^4 \sqrt{1+(y')^2}\, dx = \int_0^4 \sqrt{1+\frac{9}{4}x}\, dx \underset{\downarrow}{=} \frac{4}{9}\int_1^{10} \sqrt{u}\, du$

$u = 1+\frac{9}{4}x$ $\quad x=0 \leadsto u=1$
$du = \frac{9}{4}dx$ $\quad x=4 \leadsto u=10$

$$= \frac{4}{9}\,\frac{2}{3}\, u^{3/2}\Big|_1^{10} = \frac{8}{27}\left(10^{3/2}-1\right).$$

Soln 2: Use implicit differentiation to find y'

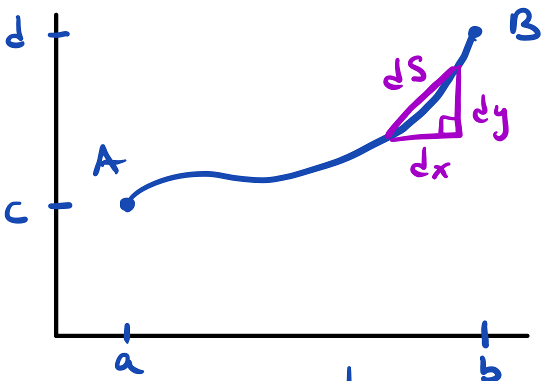
$$2yy' = 3x^2$$

For $y \neq 0$ we get $\quad y' = \frac{3}{2}\,\frac{x^2}{y}$
(1 bad pt is ok)

$$(y')^2 = \frac{9}{4}\,\frac{x^4}{y^2} = \frac{9}{4}\,\frac{x^4}{x^3} = \frac{9}{4}x$$

We get $\quad L = \int_0^4 \sqrt{1+\frac{9}{4}x}\, dx = \frac{8}{27}\left(10^{3/2}-1\right)$ (same calculation!)

Arc length element:



$$dS = \sqrt{(dx)^2+(dy)^2}$$

$$\begin{cases} dS = dx\,\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+(y')^2}\,dx \\ dS = dy\,\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+(x')^2}\,dy \end{cases}$$

So $L_{AB} = \int_a^b dS = \int_a^b \sqrt{1+(y')^2}\,dx$

$= \int_c^d dS = \int_c^d \sqrt{1+(x')^2}\,dx$ (if $y=y(x)$ can be written as $x=x(y)$.)

Q What if we write $x=x(y)$ in EXAMPLE 1?

$x = y^{2/3} \leadsto x' = \frac{2}{3}y^{-1/3}$ so $(x')^2 = \frac{4}{9}y^{-2/3}$

$L = \int_0^8 \sqrt{1+\frac{4}{9}y^{-2/3}}\, dy \leadsto$ this is harder to compute, but it can be done!

$$\sqrt{1+\frac{4}{9}y^{-2/3}} = \sqrt{1+\frac{4}{9y^{2/3}}} = \sqrt{\frac{1}{y^{2/3}}\left(y^{2/3}+\frac{4}{9}\right)} = y^{-1/3}\sqrt{y^{2/3}+\frac{4}{9}}$$

$u = \frac{4}{9} + y^{2/3}$

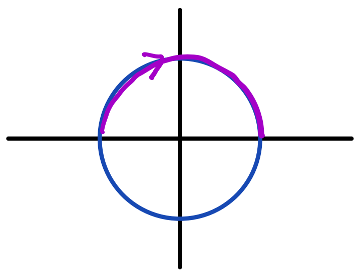
$du = \frac{2}{3}y^{-1/3}\,dy$

$y=0 \rightsquigarrow u = 4/9$

$y=8 \rightsquigarrow u = \frac{4}{9}+4 = \frac{40}{9}$

So $$L = \int_{4/9}^{40/9} \sqrt{u}\,\frac{3}{2}\,du = \frac{8}{27}\left(10^{3/2}-1\right)$$

EXAMPLE 2: Circumference of radius r



Top-half can be parametrized:

$f(x) = \sqrt{r^2-x^2}$ on $-r \le x \le r$ $\rightsquigarrow$ $f'(x) = \frac{1}{2}\,\frac{-2x}{\sqrt{r^2-x^2}} = \frac{-x}{\sqrt{r^2-x^2}}$

$$(f'(x))^2 = \frac{x^2}{r^2-x^2}$$

$$L = 2\int_{-r}^{r}\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx = 2\int_{-r}^{r}\frac{r}{\sqrt{r^2-x^2}}\,dx = 2\int_{-r}^{r}\frac{r}{r\sqrt{1-\left(\frac{x}{r}\right)^2}}\,dr$$

$$= 2\int_{-1}^{1}\frac{r}{\sqrt{1-u^2}}\,du = r\underbrace{\left(2\int_{-1}^{1}\frac{1}{\sqrt{1-u^2}}\,du\right)}$$

$u = \frac{x}{r}$

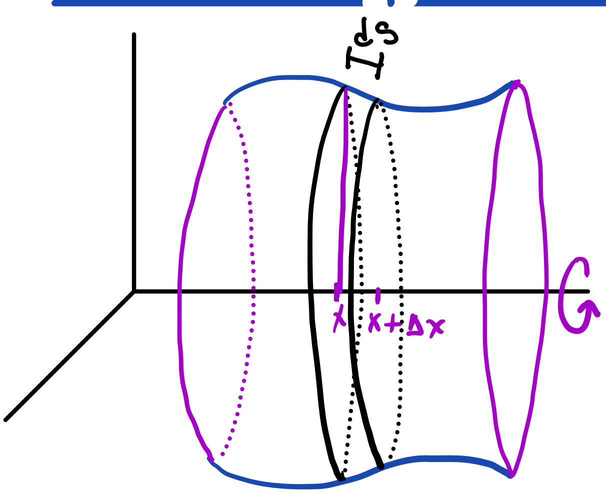
$du = \frac{1}{r}\,dx$

$x=-r \rightsquigarrow u=-1$

$x = r \rightsquigarrow u = 1$

circumference of unit disk (we'll compute it later with trig substitutions!)

§2 Surfaces of Revolution:

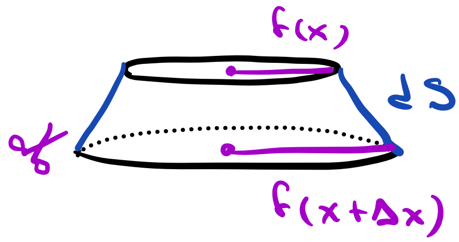


INPUT: $f:[a,b] \longrightarrow \mathbb{R}_{\ge 0}$ continuous and positive

PROCESS: Revolve $y=f(x)$ about the x-axis & take the surface S of the solid of revolution (S = surface of revolution)

Q: What's Area(S)?

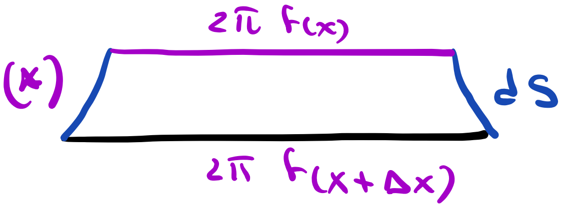
STEP 1 Take n strips $[x_{k-1}, x_k]$ for $k = 1, \dots, n$ & take the surfaces of revolution obtained from f on each strip. We get the Frustum (of a cone)



$$\text{Area(Frustum)} \overset{(*)}{\approx} 2\pi f(x)\, dS$$

$$= 2\pi f(x) \sqrt{1+(f')^2}\, dx$$

need f' to be continuous



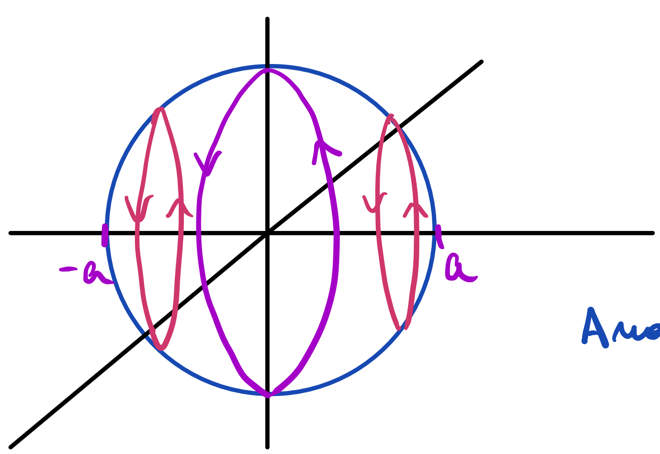
$\approx$ trapezoid so $\text{Area} \approx dS \cdot 2\pi \dfrac{(f(x) + f(x+\Delta x))}{2}$

STEP 2 Compute Riemann Sums

$$\text{Area}(S) = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \text{Area Frustum on } [x_{k-1}, x_k]$$

gives
$$\boxed{\text{Area}(S) = \int_a^b 2\pi f(x) \sqrt{1+(f'(x))^2}\, dx}$$
if f' is continuous

EXAMPLE 1: Sphere of radius a:



$$f(x) = \sqrt{a^2 - x^2}$$

$$f' = \frac{-x}{\sqrt{a^2-x^2}} \quad \text{so } (f')^2 = \frac{x^2}{a^2-x^2}.$$

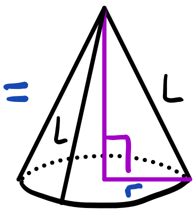
$$\text{Area}(S) = \int_{-a}^{a} 2\pi \sqrt{a^2-x^2} \sqrt{1 + \frac{x^2}{a^2-x^2}}\, dx$$

$$= \int_{-a}^{a} 2\pi a\, dx = 2\pi a (2a) = 4\pi a^2$$

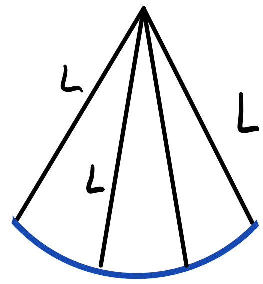
Optional Reading: **Q**: Why $(*)$?

A: Use the cone model! Cutting the frustum gives a "bent trapezoid".

• **Model Example** : Cone

$S =$ 

We cut it open & lay it out = circular sector of a disk of radius L



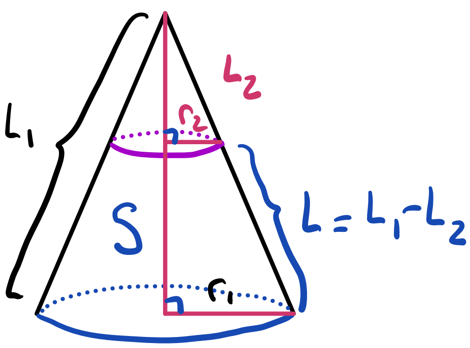
Total area$(\overset{L}{\ominus}) = \pi L^2 \rightsquigarrow$ Sector area $= \pi L^2 \left(\frac{\alpha}{2\pi}\right)$

Circumference of the sector $= 2\pi L \frac{\alpha}{2\pi}$ = full circumference of base of the cone $\overset{\parallel}{2\pi r}$

So $\frac{L\alpha}{2\pi} = r$, giving $\boxed{\frac{r}{L} = \frac{\alpha}{2\pi}}$

Conclusion : $\text{Area}(S) = \pi L^2 \frac{r}{L} = \pi r L$

• **Frustum of a cone** : We start from L, r_1, r_2 & complete it to get a full cone of side length L_1, with a base of radius r_1.



$$\begin{aligned}\text{Area}(S) &= \text{Area Cone}(L_1, r_1) - \text{Area Cone}(L_2, r_2)\\ &= \pi r_1 L_1 - \pi r_2 L_2\\ &= \pi(r_1 L_1 - r_2 L_2) \quad (**)\end{aligned}$$

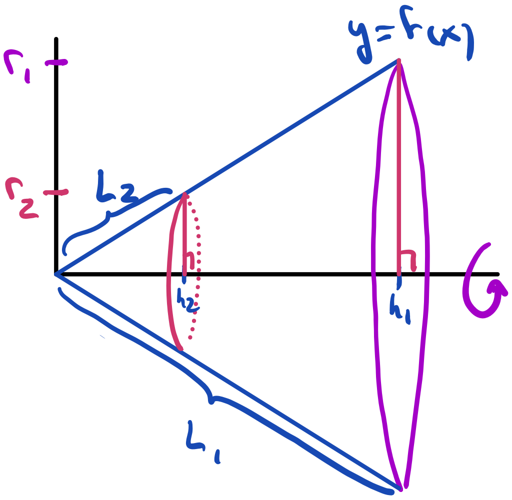
• But similarity of Δ gives $\frac{L_2}{r_2} = \frac{L_1}{r_1}$ so $r_2 L_1 = r_1 L_2$

• Now, we add and substract $\pi r_1 L_2$ in $(**)$ to get:

$$\begin{aligned}\text{Area}(S) &= \pi(r_1 L_1 - r_1 L_2 + \underbrace{r_1 L_2}_{= r_2 L_1} - r_2 L_2)\\ &= \pi(r_1(L_1 - L_2) + r_2(L_1 - L_2)) = \pi(r_1 + r_2)(L_1 - L_2)\end{aligned}$$

Conclude : $\text{Area (Frustum)} = 2\pi \left(\frac{r_1 + r_2}{2}\right) L$ (predicted by $(*)$)

EXAMPLE 2 We verify this formula with our integral formula:



Cone = solid of revolution obtained by rotating a segment.

$$y = f(x) = mx + b$$

At $(0,0)$: $\quad 0 = 0 + b \quad$ so $\quad b = 0$

At (h_2, r_2): $\quad r_2 = m h_2 \quad$ so $\quad f(x) = \frac{r_2}{h_2} x$

Then $f'(x) = \frac{r_2}{h_2}$ continuous on $[0, h_1]$

$$\text{Area(Frustum)} = \int_{h_2}^{h_1} 2\pi \frac{r_2}{h_2} x \sqrt{1 + \left(\frac{r_2}{h_2}\right)^2}\, dx$$

$$= \int_{h_2}^{h_1} 2\pi \frac{r_2}{h_2} x \frac{\sqrt{h_2^2 + r_2^2}}{h_2}\, dx$$

$$= \pi \frac{r_2}{h_2^2} x^2 \sqrt{h_2^2 + r_2^2}\ \Big|_{h_2}^{h_1} = \frac{\pi r_2}{h_2^2} \sqrt{h_2^2 + r_2^2}\,(h_1^2 - h_2^2)$$

In order to get $2\pi \frac{(r_1 + r_2)}{2}(L_1 - L_2)$, we need to use trigonometry

By similarity, we get $\frac{h_2}{h_1} = \frac{L_2}{L_1} = \frac{r_2}{r_1}$, $\quad \sqrt{h_1^2 + r_1^2} = L_1$, $\quad \sqrt{h_2^2 + r_2^2} = L_2$

So $\frac{\pi r_2}{h_2^2} \sqrt{h_2^2 + r_2^2}\,(h_1^2 - h_2^2) = \pi r_2 L_2 \left(\left(\frac{h_1}{h_2}\right)^2 - 1\right) = \pi r_2 L_2 \left(\left(\frac{L_1}{L_2}\right)^2 - 1\right)$

$$= \pi r_2 \frac{1}{L_2}(L_1^2 - L_2^2) = \frac{\pi r_2}{L_2}(L_1 - L_2)(L_1 + L_2)$$

$$= \pi (L_1 - L_2)\left(r_2 \frac{L_1}{L_2} + r_2\right)$$

$$= \pi (L_1 - L_2)\left(r_2 \cdot \frac{r_1}{r_2} + r_2\right)$$

$$= \pi (L_1 - L_2)(r_1 + r_2)$$

Conclude: Area(Frustum) $= \pi (r_1 + r_2)(L_1 - L_2)$ as we wanted!