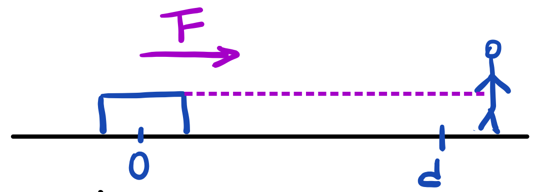
Lecture XXIX: § 7.7 Work & Energy

§1. Work:

Definition: Work is the effort done by a force to move an object, when the action is in the direction of the movement.

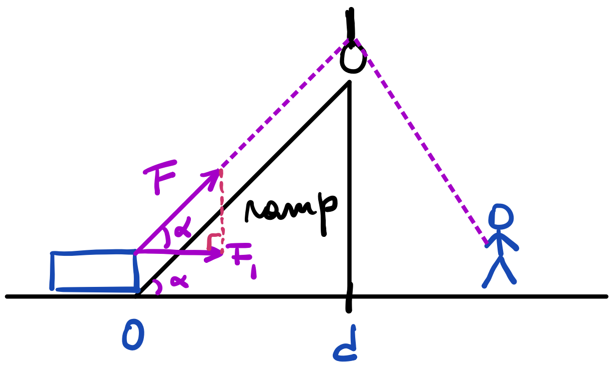
EXAMPLES: ①



- Pull an object with a chord
- F = force, d = distance

⇝ $\boxed{W = F \cdot d}$

②



- Pull an object through a pulley on a ramp.
- Component of F in the direction of the movement is $F_1 = F\cos\alpha$

⇝ $\boxed{W = (F\cos\alpha)d}$

Units: • $F_{(t)} = m \cdot a_{(t)}$ $\quad a_{(t)} = s''(t)$ ⇝ $kg \cdot \frac{m}{s^2}$, $lb\ \frac{in}{s^2}$

$\left(\text{mass} \cdot \frac{\text{distance}}{(\text{Time})^2}\right)$

• $W = F \cdot d$ ⇝ $kg\ \frac{m^2}{s^2}$, $lb\ \frac{in^2}{s^2}$ $\quad \left(\text{mass} \cdot \frac{\text{distance}^2}{(\text{Time})^2}\right)$

Example: Weight = measure of force with which an object is attracted to Earth.

Work = ft · pounds = $\boxed{\text{ft-lb}}$

• If we lift a box weighing 20 lbs, 3 ft high, the work done is 60 ft-lbs.

Other unit systems: ① cgs (cm-grams-second) ⇝ F in dynes = $1\frac{cm\,gr}{s^2}$

② mks (m-kg-second) ⇝ F in Newtons

$1N = 1\frac{m\,kg}{s^2}$

⇝ Work in cgs = dyne-cm =: erg

Work in mks = N-m =: Joule (J)

Conversion: 1 ft-lb $= 1.356$ J , 1 J $= 10^7$ ergs

Examples above: The force is the same no matter the location of the object.

Q: What if the force is non-constant? $\rightsquigarrow$ $F = F_{(x)}$ changes with x = distance.

A: Over a small distance Δx, we can "pretend" F is constant (by continuity). So:

$$\Delta W = F\,\Delta x = F\,dx \rightsquigarrow \boxed{W_{(u)} = \int_0^u F_{(x)}\,dx}$$

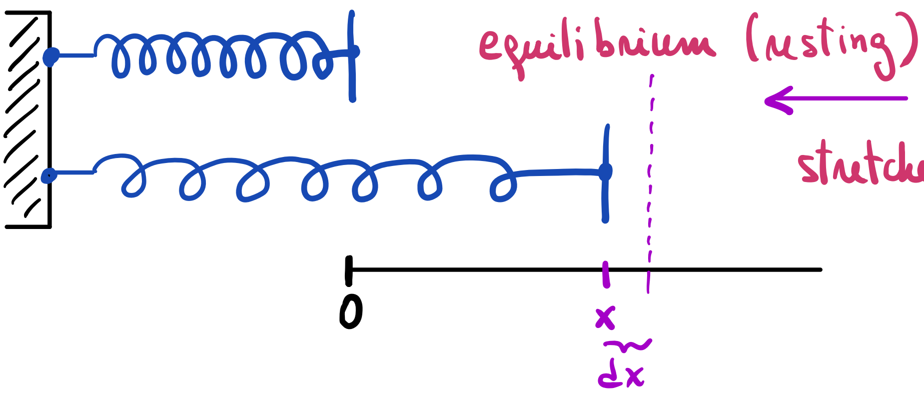
Linear approximation $\rightarrow$ $\Delta W \approx dW$

Fundamental Theorem of Calculus

for u in $\mathbb{R}$

§2. Prototypical example 1: SPRING.

- As we stretch a spring, the force pushes it back to equilibrium
- More force acts the further we stretch the spring.



$\leftarrow$ F = restorative force (against the stretching direction!)

Hooke's Law: $F_{(x)} = k\,x$ where k = spring constant (k depends on the material & properties of the spring).

Example: Assume it takes 8 lb of force to hold a 16 in long spring 2 in away from equilibrium. How much work is done when moving the spring to a 24 in length?

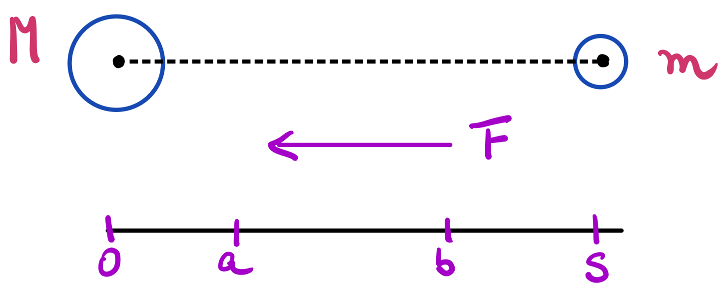
Solution: • We use the data to find k: $F_{(2)} = k \cdot 2\text{ in} = 8\text{ lb}$

So $\boxed{k = 4\ \frac{lb}{in}}$

- x = distance = $24-16=8$

So $W = \left(\int_0^8 4x\,dx\right)\frac{lb}{in} = 2x^2\Big|_0^8 \frac{lb}{in} = 128\ in^2\,\frac{lb}{in} = \boxed{128\ \text{in-lb}}$

§3. Prototypical example 2: GRAVITY.



- Replace each planet by a heavy center of mass (of masses M & m)
- Larger body is the attractor

The force of attraction of 2 bodies of mass M & m with $M>m$ is $F(s) = - \dfrac{G\cdot M\cdot m}{s^2}$ $G>0$ universal gravity constant ($G = 6.670\cdot 10^{-11}\ N\cdot\frac{m^2}{kg^2}$)

opposite from distance measurement

So $dW = F(s)\,ds = -G\dfrac{Mm}{s^2}\,ds$

Then, moving m from $s=a$ to $s=b$ (against the force of gravity) requires:

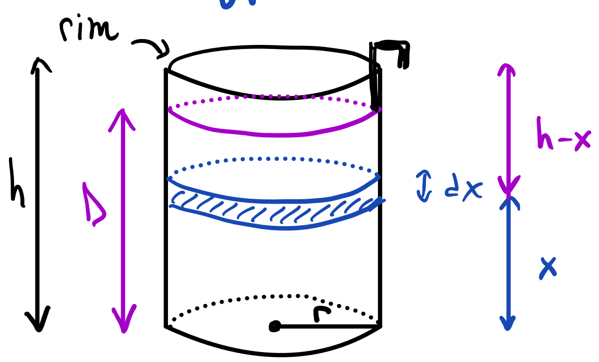
$$W = -\int_a^b F(s)\,ds = \int_a^b G\frac{Mm}{s^2}\,ds = GMm\left(\frac{-1}{s}\right)\Big|_a^b$$

$$= GMm\left(\frac{-1}{b}+\frac{1}{a}\right) = GMm\left(\frac{1}{a}-\frac{1}{b}\right) = \begin{cases} >0 & \text{if } b>a \\ <0 & \text{if } b<a \end{cases}$$

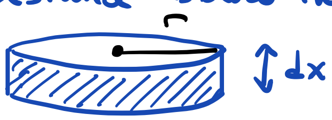
Remark: If $b\to\infty$, then $W \xrightarrow[b\to\infty]{} \dfrac{GMm}{a} > 0$

This is the work required to completely separate the 2 objects.

§4. Prototypical example 3: PUMPING WATER from a tank



- Cylindrical tank of radius $= r$ & height $= h$
- The tank is filled with water to depth D
- The body to move are drops of water
- w = weight-density of water = weight/unit vol $\left(\text{lb}/\text{in}^3\right)$

- **Q:** How much work is done to pump the water over the rim of the tank?
- **Key assumption:** The work is the same for all drops of water at the same distance below the rim (so it only depends on $h-x$)
- 

  $\text{Vol} = \pi r^2 dx$

  Thus, the weight of this is computed as $F(x) = \text{density} \cdot \text{Vol} = w\pi r^2 dx$
- Since we travel distance $(h-x)$ to pump water out, we get:

$$dW = F_{(x)}(h-x) = w\pi r^2 (h-x)\,dx$$

Conclude: $$W = \int_0^D w\pi r^2 (h-x)\,dx = w\pi r^2 \left(hx - \frac{x^2}{2}\right)\Big|_0^D = \boxed{w\pi r^2 D\left(h - \frac{D}{2}\right)}$$

Observe: Measuring from the top gives the same result.

Set $\tilde{x} = h-x$, so $d\tilde{x} = -dx$ & we flip the limits of integration

$$W = \int_{h-D}^{h} w\pi r^2 x\,dx = w\pi r^2 \frac{x^2}{2}\Big|_{h-D}^{h} = \frac{w\pi r^2}{2}\left(h^2 - (h-D)^2\right) = \frac{w\pi r^2}{2} D(2h-D)$$

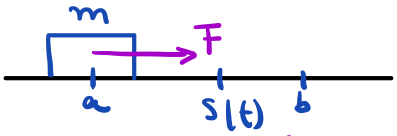
§5. Energy:

Energy comes in 2 flavors: (1) Kinetic Energy

(2) Potential Energy

① KINETIC ENERGY: energy due to motion

It's value is $\frac{1}{2} m v^2$.



$F = m\,a(t) = m\dfrac{dv}{dt}$ and $v = \dfrac{ds}{dt}$ = velocity

(F move the object along the straight line!)

<u>Theorem</u>: The work done by F equals the change in the kinetic energy of the particle.

<u>Proof</u>: We use the Chain Rule to write $F = m\dfrac{dv}{dt} = m\dfrac{dv}{ds}\dfrac{ds}{dt} = mv\dfrac{dv}{ds}$

(this is the same idea we used for computing escape velocity)

Then: $W = \displaystyle\int_a^b F(s)\,ds = \int_a^b mv\frac{dv}{ds}\,ds \underset{\text{change from } s \text{ to } v}{=} \int_{v_a}^{v_b} mv\,dv = \frac{1}{2}mv^2\Big|_{v_a}^{v_b} = \frac{1}{2}m(v_b^2 - v_a^2)$. ☐

② POTENTIAL ENERGY: energy held by an object because of its position, relative to the objects, stresses within itself, its electric charge, etc.

Example: (1) gravitational potential energy of an extended spring.

(2) electric potential energy of an electric charge in an electric field.

Potential energy is associated with forces that act on a body in such a way that the total work done by these forces on the body depends <u>only</u> on the initial & final position of the body.

<u>Conclude</u>: Potential energy $= V(s)$ = antiderivative of $-F(s) = -\int F(s)\,ds$

So $V(s) = -F(s)\,ds$

Then $W(a \to s) = -\big(V(s) - V(a)\big) = V(a) - V(s)$

(work done by the force when moving from a to s)

By Theorem, we get $W(a \to s) = \frac{1}{2}m(v_s^2 - v_a^2)$

<u>Conclude</u>: $V(a) - V(s) = \frac{1}{2}m(v_s^2 - v_a^2)$

Regrouping yields: $\underbrace{V(a) + \frac{1}{2}mv_a^2}_{\text{independent of } s(t)!} = \overbrace{V(s)}^{\text{potential energy}} + \underbrace{\frac{1}{2}mv_s^2}_{\text{kinetic energy}} = \text{Energy}$

<u>Law of conservation of Energy</u>: The total energy of the particle is conserved at every step of the movement.

<u>Nice application</u>: work of a single heart stroke (pump) is $W = 0.74$ ft-lb.