Lecture XXX : §8.2 Exponents & Logarithms

§1 Basics on exponents:

Q: Given $a>0$ & x in $\mathbb{R}$, what is a^x?

- We know the answer for special cases

① If x is an integer:

1) $x = n > 0$ $\qquad a^n = \underbrace{a \cdot a \cdots a}_{n \text{ times}}$

2) $x = 0$ $\qquad a^0 = 1 \quad (\text{for } a \neq 0)$

3) $x = -n < 0$ $\qquad a^{-n} = (a^{-1})^n = \underbrace{\frac{1}{a} \cdots \frac{1}{a}}_{n \text{ times}}$

② If x is rational, write $x = \frac{m}{n}$ with $n>0$ & m,n coprime $\left(\frac{2}{4} = \frac{1}{2}\right)$

Then $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$, where $\sqrt[n]{a}$ is the unique positive number b with $b^n = a$ (ok for $a>0$)

Observe: We don't really need to use the reduced expression of x

$$a^{\frac{km}{kn}} = (\sqrt[kn]{a})^{km} = (c)^{km}$$ where c is the unique positive number with $c^{kn} = a$

But if $b^n = a$, then $c^k = b$ by uniqueness of c & b

Then $c^{km} = \underbrace{c \cdot c \cdots c}_{km \text{ times}} \underset{\text{regroup}}{=} \underbrace{\underbrace{c \cdots c}_{k \text{ times}} \cdots \underbrace{c \cdots c}_{k \text{ times}}}_{m \text{ times}} = (c^k)^m = b^m$

So $a^{\frac{m}{n}} = b^m = a^{\frac{km}{kn}}$ for any k integer.

⇝ General definition: Approximate x using rational numbers arbitrary close to x & define a^x via a limiting process.

Definition: $\boxed{a^x = \lim_{\substack{r \to x \\ r \text{ rational}}} a^r}$ for $a>0$ & x any real number

GOOD NEWS: By construction, $f_{(x)} = a^x$ will automatically be continuous

BAD NEWS: It's not clear $f(x)$ exists (the limit may depend on how we choose $r \to x$) For example, if $x = \frac{m}{n}$ is rational, does the limit exist and agree with the old definition $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$?

Of course, it all works out fine in the end (reason: law of exponents!)

Law of exponents:

(1) $a^x a^y = a^{x+y}$ (2) $a^{x-y} = \frac{a^x}{a^y}$ (3) $(a^x)^y = a^{x \cdot y}$

Proof: It's enough to check these formulas for x, y rational. For general x, y, approximate by rationals & take limits

Indeed, if $x = \lim r$, $y = \lim s$ (in $\mathbb{Q}$), then $x+y = \lim r+s$, $x-y = \lim r-s$, $x \cdot y = \lim \underbrace{r \cdot s}_{\text{in } \mathbb{Q}}$, & the formulas obtained by limits on both sides of (1), (2) & (3) will agree

Eg: $a^x \cdot a^y = \left(\lim_{\substack{r \to x \\ r \text{ in } \mathbb{Q}}} a^r\right)\left(\lim_{\substack{s \to y \\ s \text{ in } \mathbb{Q}}} a^s\right) \underset{\substack{\uparrow \\ \text{Limit} \\ \text{Laws}}}{=} \lim_{\substack{r \to x \\ s \to y \\ r, s \text{ in } \mathbb{Q}}} a^r a^s = \lim_{\substack{r+s \to x+y \\ (r+s) \text{ in } \mathbb{Q}}} a^{r+s} = a^{x+y}$)

Now, assume $x = \frac{m}{n}$, $y = \frac{l}{n}$ (can use the same denominator!)

Pick $b>0$ with $b^n = a$ (ie $b = \sqrt[n]{a}$)

- $a^x = (\sqrt[n]{a})^m = b^m$
- $a^y = (\sqrt[n]{a})^l = b^l$

so $a^x a^y = b^m b^l = \underbrace{b \cdots b}_{m \text{ times}} \underbrace{b \cdots b}_{l \text{ times}} = \boxed{b^{m+l}}$

(1) $a^{x+y} = a^{\frac{l+m}{n}} = (\sqrt[n]{a})^{l+m} = \boxed{b^{l+m}}$

same value ✓

② • $a^{x-y} = a^{\frac{l-m}{n}} = (\sqrt[n]{a})^{l-m} = b^{l-m}$ same value!

• $\dfrac{a^x}{a^y} = \dfrac{b^m}{b^l} = \underbrace{b\cdots b}_{m\text{ times}} \ \underbrace{\frac{1}{b}\cdots\frac{1}{b}}_{l\text{ times}} = b^{l-m}$

③ • $a^{xy} = a^{\frac{lm}{n^2}} = (\sqrt[n^2]{a})^{lm} = c^{lm}$ with $c^{n^2} = a$

But $c^{n^2} = \underbrace{c\cdots c}_{n^2\text{ times}} = \underbrace{\underbrace{(c\cdots c)}_{n\text{ times}}\cdots\underbrace{(c\cdots c)}_{n\text{ times}}}_{n\text{ times}} = a = \underbrace{b\cdots b}_{n\text{ times}}$,

so $c^n = b$.

$(a^x)^y = (b^m)^{\frac{l}{n}} = (\sqrt[n]{b^m})^l = d^l$ with $d^n = b^m$

But $d^n = b^m = (c^n)^m = \underbrace{c^n\cdots c^n}_{m\text{ times}} = c^{n\cdot m} = (c^m)^n$

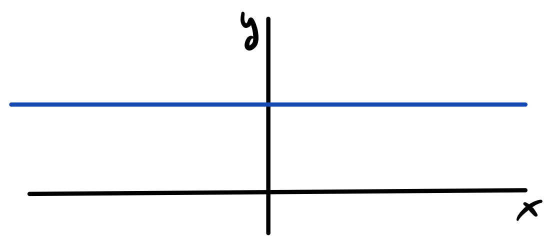
& both $d, c > 0$ so $d = c^m$.

Conclude: $a^{xy} = c^{lm} = (c^m)^l = d^l = (a^x)^y$

§2 Graphs of a^x:

We will have 3 cases: $a=1$, $a>1$ & $0<a<1$.

① If $a=1$: $a^x = 1$ for all x



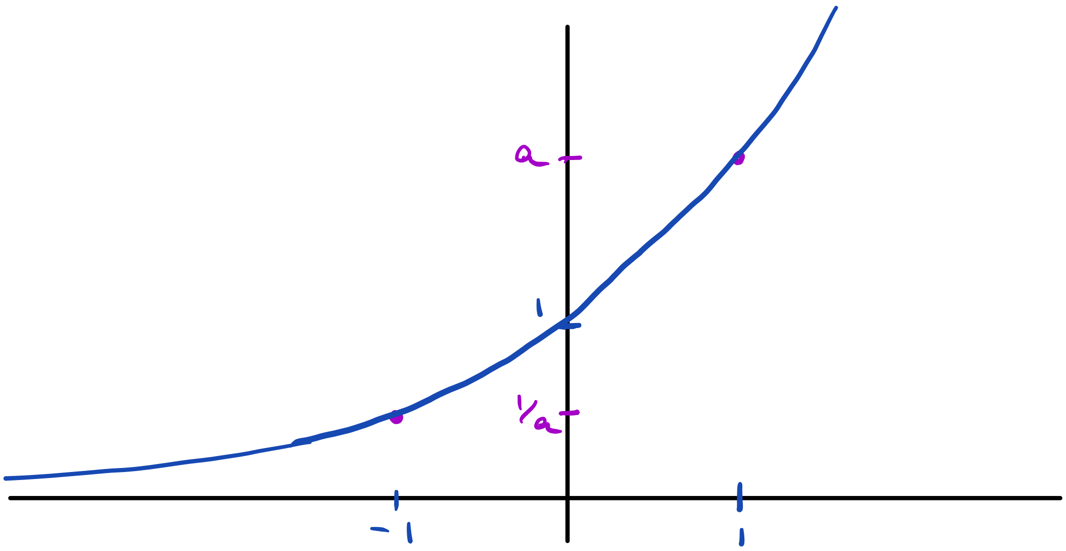
② If $a>1$: a^x is continuous & $a^r > 1$ for any rational $r>0$

so the same is true for a^x ($a^x > 1$ for $x > 0$)

• a^x is increasing (if $x<y$, then $a^y = a^{(y-x)+x} =$

$= a^{(y-x)} a^x \underset{y-x>0}{>} 1 \cdot a^x = a^x$)

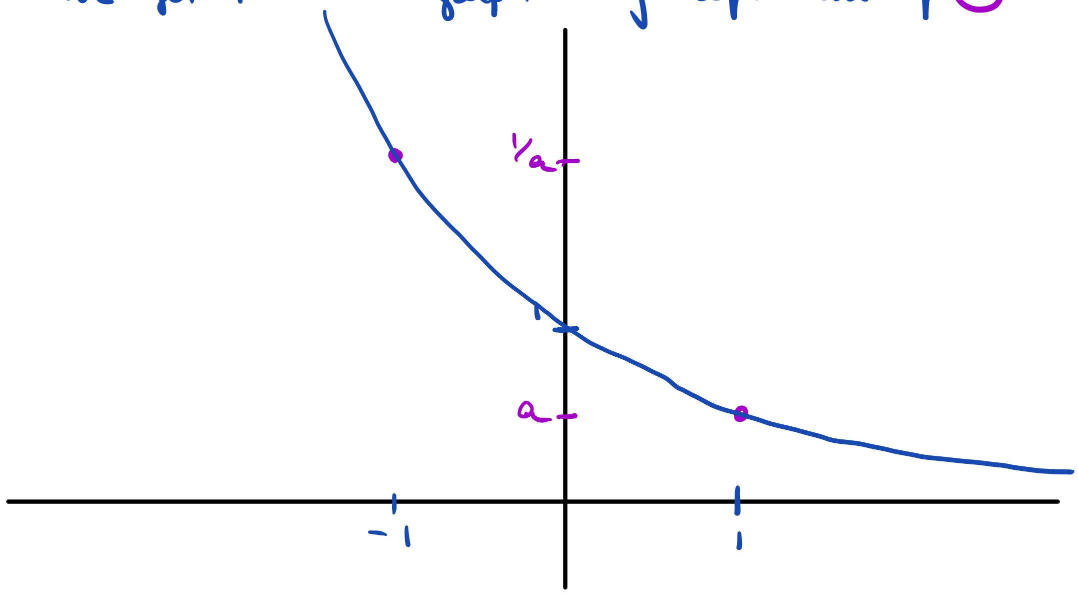
- $\lim_{x\to\infty} a^x = \infty$, $\lim_{x\to-\infty} a^x = \lim_{x\to\infty} \frac{1}{a^x} = 0$



- Range $= \{y>0\}$
- strictly increasing, so it's injective (horizontal line test)

③ <u>If $0<a<1$:</u> Write $a^x = \left(\frac{1}{a}\right)^{-x}$ & now $\frac{1}{a}>1$.

We get the new graph by reflection of ② about the y-axis



- $a^x > 1$ for $x<0$
- strictly decreasing (so injective!)
- $\lim_{x\to\infty} a^x = 0$
- $\lim_{x\to-\infty} a^x = \infty$
- Range $= \{y>0\}$

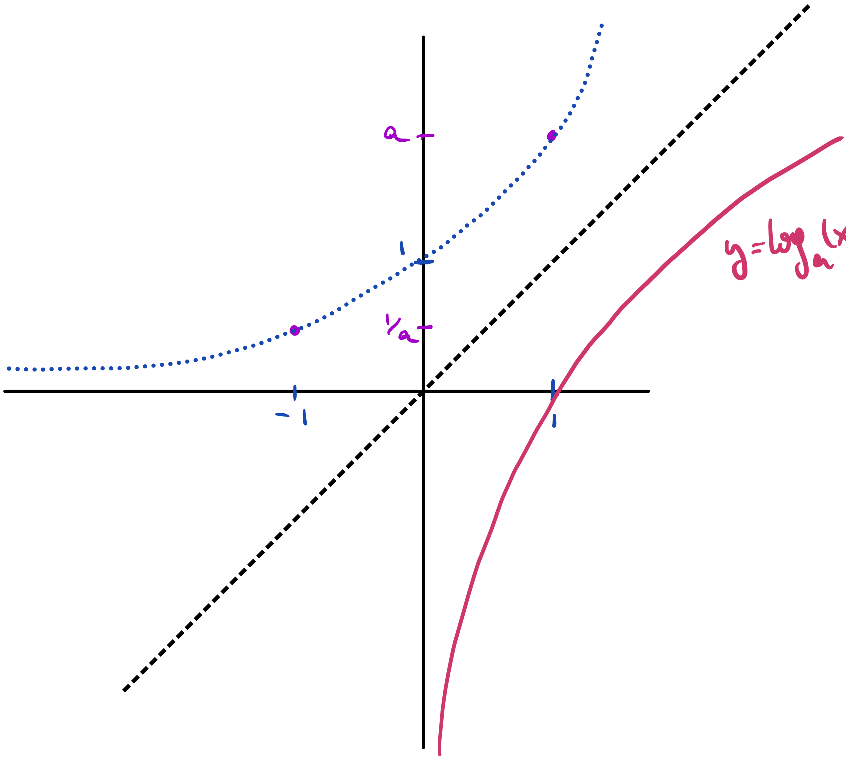
§3 Logarithms:

<u>Conclusion</u>: For $a \neq 1$, we get $a^x : \mathbb{R} \longrightarrow \{y>0\}$ is injective and surjective, so 1-to-1 & we can find an inverse function: $\log_a : \{x>0\} \longrightarrow \mathbb{R}$

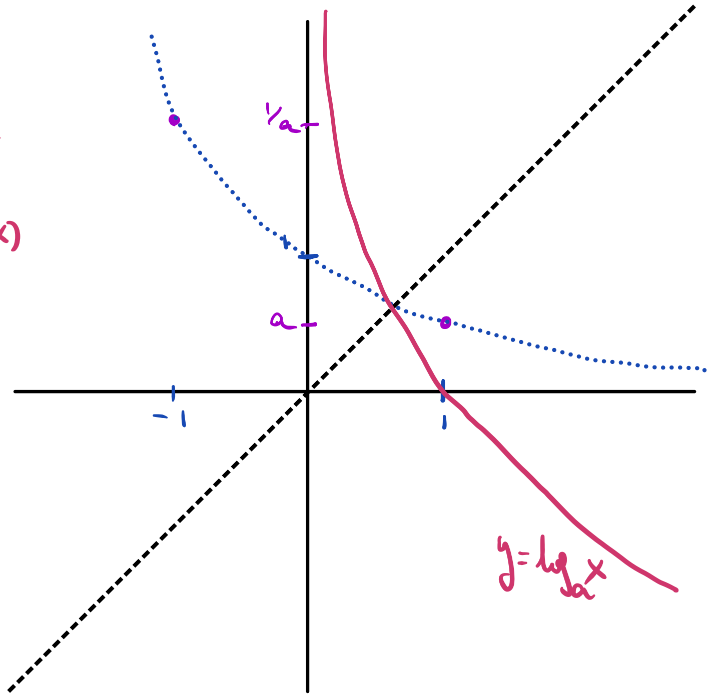
$$x \longrightarrow \log_a x$$

We call it <u>logarithm to the base a</u>

• **Graphs of $\log_a x$**: Reflect the graph of a^x along the $x=y$ line



$\boxed{a>1}$



$\boxed{0<a<1}$

Note: $\log_{1/a}(x) = -\log_a x \qquad \left[\ (a^y)=x = \left(\frac{1}{a}\right)^{-y}\ \right]$

Note: • If $a>1$ $\quad \lim_{x\to\infty} \log_a x = \infty, \quad \lim_{x\to 0} \log_a x = -\infty$

• If $0<a<1$ $\quad \lim_{x\to\infty} \log_a x = -\infty \quad \lim_{x\to 0} \log_a x = +\infty$

• **Basic properties of $\log_a$**:

① $\log_a(x_1x_2) = \log_a x_1 + \log_a x_2$.

Why? $y_1 = \log_a x_1$ means $a^{y_1} = x_1$

$y_2 = \log_a x_2$ ——— $a^{y_2} = x_2$

& $x_1x_2 = a^{y_1}a^{y_2} = a^{y_1+y_2}$ gives $y_1+y_2 = \log_a(x_1x_2)$

② $\log_a\left(\frac{x_1}{x_2}\right) = \log_a x_1 - \log_a x_2$

Why? $a^{y_1-y_2} = \frac{a^{y_1}}{a^{y_2}} = \frac{x_1}{x_2}$ so $y_1-y_2 = \log_a\left(\frac{x_1}{x_2}\right)$.

③ $\log_a(x^b) = b\log_a x$

Why? If $x = a^y$ Then $x^b = (a^y)^b = a^{by}$: so $\log_a x^b = by$

④ $\log_a(a^x) = x$ by definition ($a^x = a^{\square}$ forces $\square = x$) $\leftarrow \log_a a^x$

⑤ $a^{\log_a x} = x$ by definition ($y = \log_a x$ means $a^y = x$)

⑥ $\log_a 1 = 0$, $\log_a a = 1$