Lecture XXXI: §8.3 The Number e & the function $y=e^x$
§8.4 The natural logarithm function
§8.5 Population growth

Recall: For $a>0$ & x real, we define $a^x = \lim_{\substack{r\to x \\ r \text{ in } \mathbb{Q}}} a^r$ (exponential)

Here, $a^r = a^{\frac{m}{n}} = (\sqrt[n]{a})^m$ where $\sqrt[n]{a} > 0$ is the unique positive solution to $y^n = a$

- For $a \neq 1$: $a^x : \mathbb{R} \longrightarrow \mathbb{R}_{>0} = \{y>0\}$ is invertible, with inverse

$\log_a : \mathbb{R}_{>0} \longrightarrow \mathbb{R}$ logarithm to the base a ($a^{\log_a x} = x$ & $\log_a a^x = x$)

§1. Derivatives of exponentials:

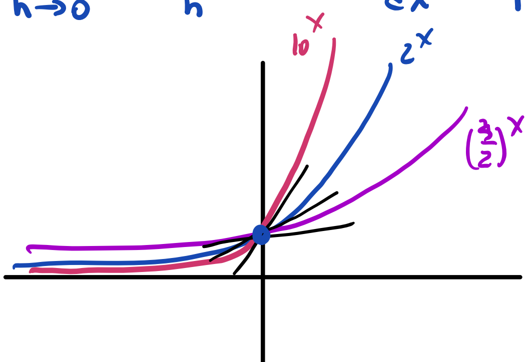
We compute $\frac{d}{dx} a^x$ for $x>0$ using the method of increments & the laws of exponents

$$\frac{d}{dx} a^x = \lim_{\Delta x \to 0} \frac{a^{x+\Delta x} - a^x}{\Delta x} = \lim_{\Delta x \to 0} \frac{a^x a^{\Delta x} - a^x}{\Delta x} = \lim_{\Delta x \to 0} a^x \frac{a^{\Delta x} - 1}{\Delta x}$$

$$= a^x \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x} \quad \text{as long as } \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x} \text{ exists!}$$

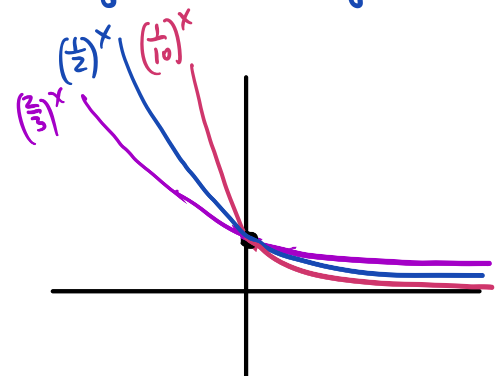
(↓ Limit Laws) (it does!)

- $\lim_{h\to 0} \frac{a^h - 1}{h} = \frac{d}{dx} a^x \Big|_{x=0}$ = slope of the tangent line to $y=a^x$ at $(0,1)$



positive slopes

$a>1$



negative slopes

$0<a<1$

Definition e is the unique number satisfying $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$ (tangent line has slope 1)

$e = 2.71828\ldots = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$ (accuracy up to 4 decimal places by setting $n = 100{,}000$)

(e for explosion) ↳ Jacob Bernoulli (1695-1705)

Euler found an alternative way to compute this limit.

$e = \lim_{n\to\infty} \left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right)$ (accuracy up to 4 decimal places by setting $n=7$)

Observation: $\frac{d}{dx} e^x = e^x$ Call e^x the exponential function

It's the unique solution to $\begin{cases} f' = f \\ f(0) = 1 \end{cases}$

- In turn $\frac{d}{dx}(ce^x) = c\frac{d}{dx}e^x = ce^x$ also solves $f' = f$.

Proposition All solutions to $f' = f$ are of the form $f(x) = ce^x$ for some parameter c.

Why? Consider a solution $f(x)$ & write $g(x) = \frac{f(x)}{e^x}$.

By the Quotient Rule we get

$$g' = \frac{f'_{(x)}e^x - f(x)(e^x)'}{(e^x)^2} = \frac{f'e^x - fe^x}{e^{2x}} = \frac{e^x\overbrace{(f'-f)}^{=0}}{e^{2x}} = 0$$

And $g'_{(x)} \equiv 0$ for all x forces g to be a constant $(=c)$

Then $\frac{f(x)}{e^x} = c$ meaning $f_{(x)} = ce^x$

Consequence: $\frac{d}{dx}e^x = e^x$ gives $\int e^x\,dx = e^x + C$

Examples: ① $\int e^{5x}\,dx \underset{\substack{u=5x \\ du=5dx}}{=} \int e^u \frac{du}{5} = \frac{1}{5}\int e^u du = \frac{1}{5}(e^u + C)$

$$= \frac{e^{5x}}{5} + \underbrace{\frac{C}{5}}_{\text{constant } = \tilde{C}} = \frac{e^{5x}}{5} + \tilde{C}$$

② $\int xe^{x^2}\,dx \underset{\substack{u=x^2 \\ du=2x\,dx}}{=} \int e^u \frac{du}{2} = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C$

§2. The natural logarithm:

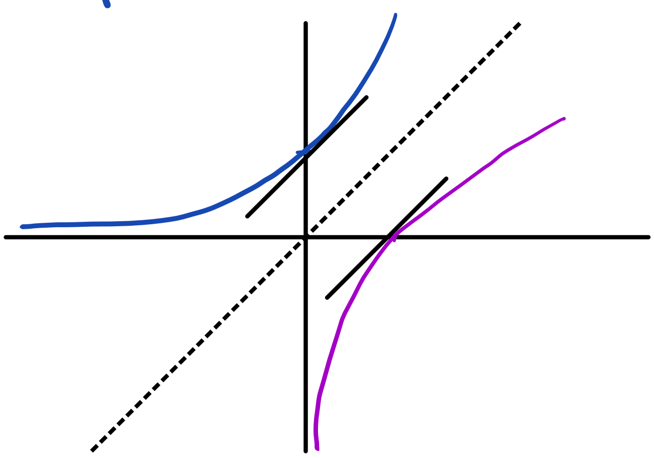
Definition: $\ln(x) = \log_e(x)$ so $y = \ln(x)$ means $e^y = x$

Proposition: $\ln(x)$ is infinitely differentiable and $\frac{d}{dx}\ln x = \frac{1}{x}$

Why? Use implicit differentiation on $e^y = x$ $\quad y = y(x)$

$\frac{d}{dx}$ on $e^y = x$ gives $e^y y' = 1$ so $y' = \frac{1}{e^y} = \frac{1}{x}$

Properties of $\ln(x)$:



- Slopes of tangent lines for e^x & $\ln x$ are related (graphs are mirrors of each other)
- $\lim\limits_{x \to 0^+} \ln x = -\infty$ $\quad \lim\limits_{x \to \infty} \ln x = \infty$ (because $e > 1$)

Integration $\int \frac{dx}{x} = \ln|x| + C$

Examples: ① $\int \frac{x^3}{x^4+1}\,dx = \int \frac{du}{4u} = \frac{1}{4}\ln u + C = \frac{1}{4}\ln(x^4+1)$

$u = x^4+1$
$du = 4x^3\,dx$

② $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int \frac{-du}{u} = -\ln u + C$

$u = \cos x$
$du = -\sin x$

$= -\ln \cos x + C$ for $\cos x > 0$

§3. Other bases of exponentials & logarithms:

- **Key fact** If $a = e^b$ then $a^x = (e^b)^x = e^{bx}$ (law of exponents)

So $\frac{d}{dx}a^x = \frac{d}{dx}e^{bx} = be^{bx} = \boxed{(\ln a)e^{bx}}$

- Similarly if $y = \log_a x$, then $x = a^y = (e^b)^y = e^{by}$

So $\ln x = by = \ln(a)\, y$ so $\boxed{\log_a x = \frac{\ln x}{\ln a}}$

In particular: $\frac{d}{dx}(\log_a x) = \frac{1}{\ln a}\frac{d}{dx}\ln x = \boxed{\frac{1}{x \ln a}}$

Q: Growth of e^x & $\ln x$?

$\frac{d}{dx} e^x = e^x > 0$ so e^x is strictly increasing

$\frac{d}{dx} \ln x = \frac{1}{x} > 0$ so $\ln x$ ______________

But how fast do they grow? (We'll use L'Hospital rule to confirm this)

① $\lim_{x \to \infty} \frac{e^x}{x^n} = +\infty$ for any positive integer n (so e^x grows faster than ANY polynomial)

② $\lim_{x \to \infty} \frac{\ln x}{x^p} = 0$ for all $p > 0$ (so $\ln x$ grows slower than ANY positive fractional power of x, eg $x^{1/2}$, $x^{1/3}$, ...)

§4. Solving Differential Equations

Proposition: $y' = ky$ for a fixed k has solutions $y = ce^{kx}$ where c is a constant

Why? Use separation of variables $\frac{dy}{dx} = ky \rightsquigarrow \frac{dy}{y} = k\,dx$

So $\int \frac{dy}{y} = \int k\,dx = kx + C$ so $y = e^{kx+C} = e^C e^{kx}$

($\int \frac{dy}{y} = \ln y$; e^C = constant c)

To check that there are no more solutions: write $g(x) = \frac{f(x)}{e^{kx}}$ where $f(x)$ solves $f' = kf$.

Then $g'(x) = \dfrac{f' e^{kx} - f k e^{kx}}{(e^{kx})^2} = \dfrac{kf e^{kx} - fk e^{kx}}{(e^{kx})^2} = 0$

So g is a constant, that is $f(x) = c e^{kx}$ for some fixed c

§5. Population growth

Basic Model: Set $N(t)$ = population at time t (e.g. bacteria)

- **Assumptions:** unlimited food, no predators, no deaths (TOY MODEL)
- **Rate of change of current population:** $\dfrac{dN}{dt} = k N(t)$ k = constant

k = percentage of population increase.

Then, $\boxed{N(t) = N_0 e^{kt}}$ where $N_0 = N(0)$ = population at time $t=0$.

- **k vs doubling time:**

Set t_d = doubling time = time it takes to double the population (in size)

Assuming $N_0 > 0$, we get

$$2N_0 = N(t_d) = N_0 e^{kt_d}$$
$$2 = e^{kt_d}$$
$$\ln 2 = kt_d \rightsquigarrow \boxed{t_d = \frac{\ln 2}{k}}$$

Consequence: $N(t+t_d) = N_0 e^{k(t+t_d)} = N_0 e^{kt+kt_d}$

$$= \underbrace{(N_0 e^{kt})}_{=N(t)} e^{kt_d} = N(t) e^{\ln 2} = 2N(t)$$

So it doesn't matter when we start counting. The doubling time is the same!

EXAMPLE 1: The number of bacteria in a culture doubles every hour. How long does it take for 1000 bacteria to produce 1 billion $= 10^9$?

Solution: $t_d = 1$ hour so $k = \dfrac{\ln 2}{t_d} = \ln 2$, $N_0 = 1000$

$\rightsquigarrow N(t) = N_0 e^{kt} = 1000\, e^{(\ln 2)t}$

We need to find t with $10^9 = 10^3\, e^{t \ln 2}$. $\rightsquigarrow 10^6 = e^{t\ln 2}$

Take ln: $6 \ln 10 = t \ln 2$ so $t = \frac{6 \ln 10}{\ln 2} \approx 15.5$ hours

EXAMPLE 2: In 1970, the world population was 3.6 billion. The Earth weighs $6586 \cdot 10^{18}$ tn. If the population increases at a rate of 2% per year & an average person weighs 120 lbs, when will the weight of all people equal the Earth's weight?

Solution: $k = \frac{2}{100}$ & $N_0 = 3.6\, 10^9$ so $N(t) = 3.6\, 10^9\, e^{\frac{2}{100}t}$

We want t so that $120\, N(t) = 6586 \cdot 10^{18} \cdot 2000$ (1 tn = 2000 lb)

$$120 \cdot 3.6\, 10^9\, e^{t/50} = 6586\, 10^{18}\, 2 \cdot 10^3 = 6586 \cdot 10^{21} \cdot 2$$

So $e^{t/50} = \frac{6586 \cdot 2 \cdot 10^{21-9}}{120 \cdot 3.6} = \frac{3298}{108}\, 10^{12}$

$\rightsquigarrow \frac{t}{50} = \ln\left(\frac{3298}{108}\, 10^{12}\right) = \ln(3298) + 12 \ln(10) - \ln(108)$

so $t = 50\left(\ln(3298) + 12\ln(10) - \ln(108)\right) = 1552.42$ years